The Impact of Silica Aerogel Insulation on Polytropic Expansion of Hydrogen in Solar Parabolic Stirling Engines

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Abstract—In 1816, Scottish engineer Robert Stirling came up with a way to harness a thermodynamic cycle known to these days as Stirling cycle to convert heat supplied to drive the polytropic expansion and contraction of an ideal working fluid to generate shaft work. Two centuries later, though the device has been engineered to achieve up to 40% thermal efficiency despite the high temperature differentials required to operate, it never reached mass adoption, due to technical limitations for automotive applications such as weight, and the economic bottlenecks that arise for power generation applications. Within this portfolio, Silica Aerogel was investigated as an insulator. By using Fourier's law to estimate the heat source to then calculate the temperature difference between the hot and the cold side using thermal resistance a comparison of performance of the selected insulator to more traditional ones was made. The comparison was conducted by comparing the obtained ΔT to a standard temperature difference of 500 ℃ which is achieved by using cooling fins and cold water flowing through the cold side. The outcomes of the study resulted to be inconclusive due to the discrepancy in results. By using a simplified geometry, we obtained a change in temperature difference of -29% and by modelling the geometry more rigorously an increase in 829% was calculated. Nevertheless, validating the modified device could improve its chances of mass adoption in the automotive industry as Aerogel is a light insulative material, and in the energy industry as it would reduce costs.

Index Terms—Aerogel, Stirling engines, Thermal conductivity

1 Introduction

The 2nd law of thermodynamics puts a constrain to the thermal efficiency of heat engines that can be expressed by equation 1.

$$\eta \leq 1 - \frac{T_c}{T_h} \tag{1}$$

as such if $T_c \rightarrow 0$ then eta will go to 1. The temperature difference measured using commercial Stirling engines can reach 500 °C. Therefore, to calculate the % change of useful heat transferred to the system between the new and the traditional Stirling engine we can use equation 2:

$$\%change = 100 * (\frac{new}{traditional} - 1) \tag{2}$$

2 The Core Problem

2.1 Simplifying Geometry

The first step in solving the core problem consists of simplifying the geometry.

By considering only the sites that are relevant to the heat conduction that needs to be investigated we can divide the engine in two, a hot side (the head of the piston) and a cold side (the insulated side) as illustrated in Figure 2.

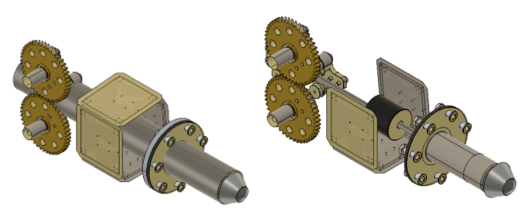

Fig. 1: Shows a model of the insulated Stirling Engine, see appendix for more detailed diagram with named sections

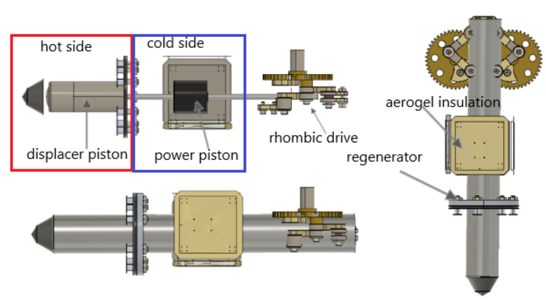

Fig. 2: Shows a schematic showing how the hot side (inside the red square) is separated by the cold side (blue square)

The geometry for the core problem will be simplified to a cylinder or a rod, and since the cold side is wider than the hot side it will be modelled as a larger cylinder as shown

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in Figure 4. For this problem we will consider x to be the

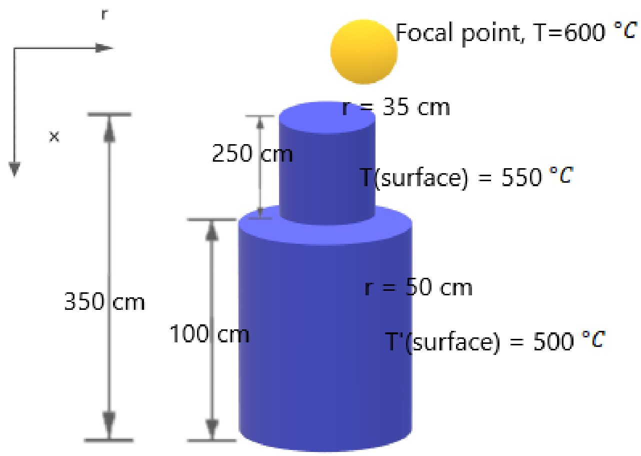


Fig. 3: Shows the geometry of the core problem with the various boundary conditions and sizes

distance form the focal point therefore moving in the same direction of the heat flow and r to be the distance from the center of the rod as can be shown in Figure **??**.

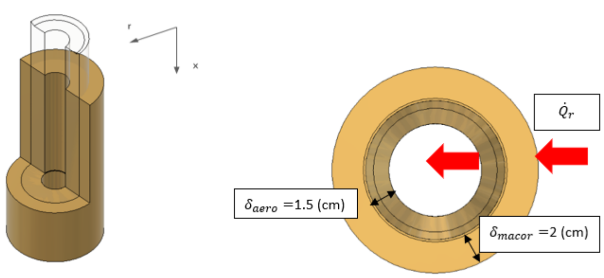


Fig. 4: Shows the coordinates of the rod

2.2 Assumptions

Constant Conductivity: Thermal conductivity depends on the material being used. This may be a function of temperature as such it may affect our integral when solving the heat equation. The three materials used for this model are:

1) **Tungsten**: in the hot side, this is because has the lowest coefficient of thermal expansion and it heats up very quickly and has high melting point. We can assume that the material has a constant thermal conductivity at high temperatures as shown in [2] and [3].
2) **Macor**: is the trademark for a machinable glass-ceramic developed and sold by Corning Inc [17]. The material will be used as insulation and is what the Aerogel is mounted on. This can be considered to have a constant thermal conductivity [16]
3) **Aerogel**: also has a constant thermal conductivity, this was reported to be 0.031 [12] at 900 degrees, and 0.01 [13] at 25 degrees. Nevertheless, regardless of the 300 prcntg increase the change is only of 0.021 k/W over the temperature range which is small.

Steady State: Solar irradiance tends to stay constant in the hourly time scale for the bulk of the day [21]. Nevertheless, since our process focuses on the time scale of seconds, we can consider the local irradiation to be constant.
Conduction is the only mode of heat transfer: For our model we are considering the hot side to be inside the focal point, thereby reducing the effect of air insulation or different forms of convection.
Heat does not move in the θ direction: Assuming that heat flows into the rod radially in an homogeneous way since the focal point is modelled as a sphere all the angles will be heated equally as such eliminating a temperature gradient.
No heat generation: since there are no reactions or sources of heat internal to the Stirling engine as the polytropic processes occur adiabatically within the hot and cold section we can assume there is no heat generation.

3 Deriving Temperature Profile

to derive the temperature profile, we can apply a heat balance on a differential element illustrated in Figure 5 . Fol-

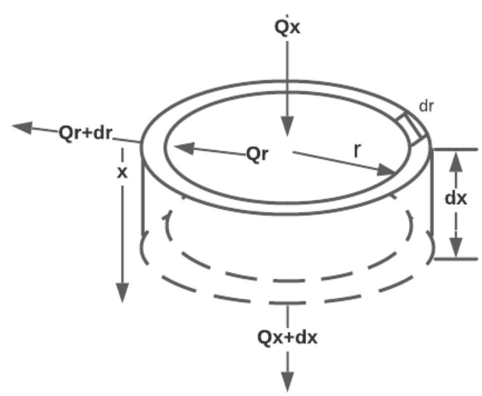


Fig. 5: Shows the heat flowing in and out of the differential element

lowing the shell balance we can obtain Laplace's equation in cylindrical coordinate for $\frac{\partial T}{\partial \theta} = 0$ as shown in equation 3:

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) = 0 \tag{3}$$

To solve the derived second order homogeneous PDE, we will make our variables dimensionless and use the following Dirichlet boundary conditions:

1) B.C. $\rightarrow T(x = 0, r) = T_{min}$
2) B.C. $\rightarrow T(x = L, r = \frac{r_{max}}{2}) = T_{max}$

the following non-dimensional parameters will be used, xi will be the negative distance from the focal point as such it will be the length of the Stirling engine :

$$\theta = \frac{T - T_{min}}{T_{max} - T_{min}} \tag{4}$$

$$\xi = \frac{-x}{L} \tag{5}$$

$$\rho = \frac{r}{r_{max}} \tag{6}$$

$$\frac{\partial^2 \theta}{\partial \xi^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho}(\frac{\partial^2 \theta}{\partial \rho^2}) = 0 \tag{7}$$

To solve this equation, we can make the following assumption:

$$\theta(\xi, \rho) = X(\xi)R(\rho) \tag{8}$$

This will make our boundary conditions become:

1) $\theta(0, \rho) = 0 = X(0)R(\rho)$
2) $\theta(1, 0.5) = 1 = X(1)R(0.5)$

Expressing the PDE in terms of X and R and by, dividing through by $\frac{RX}{L^2}$ we can see that if xi is changing for a fixed position rho then the RHS of the equation is constant , so the LHS must also be constant, we will set this constant as k squared which will be the eigenvalue of our function as shown in equation 9.

$$-\frac{1}{X}\frac{\partial^2 X}{\partial \xi^2} = -\frac{L^2}{R}(\frac{1}{\rho}\frac{\partial R}{\partial \rho} + \frac{\partial^2 R}{\partial \rho^2})) = \pm k^2 \tag{9}$$

we will take the case where our eigenvalue is positive so that we can separate the two functions and our X variable will become a particular PDE that fall into the category of Sturm-Liouville Equations.

$$\frac{\partial^2 X}{\partial \xi^2} + k^2 X = 0 \tag{10}$$

for a positive eigenvalue k squared (that is the only scenario that doesn't lead to a-physical behaviours) we get equation 11

$$X(\xi) = Asin(k^2\xi) + Bcos(k^2\xi) \tag{11}$$

The second equation instead falls into the category of Bessel functions of the form shown by equation 12:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - \alpha^2)y = 0 \tag{12}$$

In our case the value of alpha is zero, lambda represents our eigenvalue as our equation in rho variables is shown by equation 13:

$$\rho^2\frac{\partial^2 R}{\partial \rho^2} + \frac{\partial R}{\partial \rho} + \rho^2(\frac{k}{L})^2 R = 0 \tag{13}$$

The general solution to the Bessel function is shown by equation **??**

$$J_\alpha(\lambda x) = \sum_{m=0}^{\infty}(\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}(\frac{\lambda x}{2})^{2m+\alpha}) \tag{14}$$

where the gamma function is a shifted generalised factorial that can be expressed as the following integral shown in equation **??** using Laplace transform:

$$\Gamma(n) = (n-1)! = \int_0^\infty (x^{n-1}e^{-x})dx \tag{15}$$

Applying the boundary conditions to the dimensionless PDE we obtain the temperature profile shown by equation 16, where J_{mn} are roots of the bessle function J_m for different values of n. For more information on the derivation see Appendix section (The core problem – Deriving temperature profile).

$$T(x, r) = T_{min} + \Delta T\sum_{n=0}^{\infty} B(sin(\frac{-(J_{mn})^2 x}{L})J_0(j_{0,n}\frac{r}{r_{max}}) \tag{16}$$

Where B is:

$$B = \frac{1}{sin((J_{mn})^2)J_0(\frac{J_{0,n}}{2})} \tag{17}$$

4 Evaluation

Now that we have a temperature profile we can derive the temperature gradient term in the general form of Fourier's Law is shown by equation 19 to obtain the heat that goes into the cold side and equate it to the thermal resistance equation

$$\dot{Q}_{in} = \frac{\Delta T}{R_{tot}} \tag{18}$$

show by equation 18 to get ΔT .

$$q = -k\nabla T \tag{19}$$

where q is the heat flux in $\frac{W}{m^2}$ and our temperature gradient equation 20:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{\partial T}{\partial x}\hat{x} \tag{20}$$

Where $\hat{r}$, $\hat{x}$, $\hat{\theta}$ are the unit vectors in the r , x and θ direction accordingly. For an Anisotropic material the thermal conductivity k in this equation is a second order tensor or a 3x3 matrix. To find the entries to the transformation we could use a numerical algorithm as done in [18].

$$k = \begin{pmatrix} k_{rr} & k_{rx} & k_{r\theta} \\ k_{xr} & k_{xx} & k_{x\theta} \\ k_{\theta r} & k_{\theta x} & k_{\theta\theta} \end{pmatrix} \tag{21}$$

Pure tungsten is homogeneous, and its crystalline structure exhibits mainly two forms alpha or beta [19]. Nevertheless, depending on the way it has been manufactured crystallographic defects, such as vacancies, grain boundaries and dislocations may affect the thermal conductivity of tungsten as it could exhibit anisotropic behaviour[20]. However, since tungsten forms very strong metallic bonds, dislocations are unlikely to occur, as such we can assume that the thermal conductivity tensor transforms the r, x and theta components of the T vector equally by scalar k. This corresponds to an isotropic scalar k which can be represented as follows:

$$k = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \tag{22}$$

this can be simplified as follows:

$$k = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = k \times I \approx k \tag{23}$$

Where I represent the identity matrix. The result makes sense since pure metals often have a constant thermal conductivity and are isotropic.

to calculate the thermal resistance in a cylinder we can use equation 24:

$$R_{tot} = \sum \left(\frac{ln(\frac{r_o}{r_i})}{2\pi kL}\right)_i \tag{24}$$

where i refers to each layer of insulation that heat transfers through. The total thermal resistance was calculated using equation 25

$$R_{tot} = \frac{ln(\frac{r_{max}}{r_{max}-\sigma_{aerogel}})}{2\pi k_{aerogel}(L - x_{contact})} + \frac{ln(\frac{r_{max}}{r_{max}-\sigma_{macor}})}{2\pi k_{macor}(L - x_{contact})} \tag{25}$$

This gave a total thermal resistance of $\approx 0.215\frac{k}{W}$. Since we are considering the heat coming from the hot side into the cold side, the temperature gradient will be evaluated at: x=2.5m and r=0.35m. Therefore, we can calculate the r and x components of the heat flow vector as follows:

$$\dot{Q}_x = -Ak\frac{\partial T}{\partial x} = (-2\pi rL)_{cold}k_{hot}(\frac{\partial T}{\partial x})_{x=2.5,r=0.35} \tag{26}$$

$$\dot{Q}_r = -Ak\frac{\partial T}{\partial r} = (-2\pi rL)_{cold}k_{hot}(\frac{\partial T}{\partial r})_{x=2.5,r=0.35} \tag{27}$$

As we expected the x component of the heat flow vector is positive, this is because we set the x coordinates to go to the same direction of heat. Evaluating the two components we get:

$$\dot{Q}_r = 3942.68W \tag{28}$$

$$\dot{Q}_x = -158.783W \tag{29}$$

to calculate T_c now we can equate the magnitude of the heat flow coming from the hot rod to the thermal resistance equation from the cold rod:

$$|\dot{Q}| = \sqrt{(\dot{Q})^2 + (\dot{Q})^2} = 3939.48W \tag{30}$$

$$\dot{Q_{in}} = 3939.48W = \frac{\Delta T}{R_{tot}} \tag{31}$$

we get $T_c = -246.99K$ In terms of a percentage change this is:

$$\%change = 100 * (\frac{846.99}{500} - 1) = 69.4\% \tag{32}$$

5 The Complex Problem

5.1 Modelling The geometry

For the complex problem a more rigorous geometry was modelled. This was done by dividing the Stiling engine illustrated in Figure 1 of the introduction into 3 parts without considering the rhombic drive. The tip of the hot side illustrated by Figure **??**, the body of the hot side which is a rod similar to the one modelled in the previous section, and a slab that would represent the cold side. The size of the different sections with the boundary conditions is illustrated in Figure **??**: Upon visual inspection we can observe that the area of the tip increases with increasing x (distance from the top) by plotting a cross-sectional area as displayed in Figure 9 We can model the cross-sectional area of the tip as a quadratic, using equation 33

$$A = \alpha x^2 \tag{33}$$

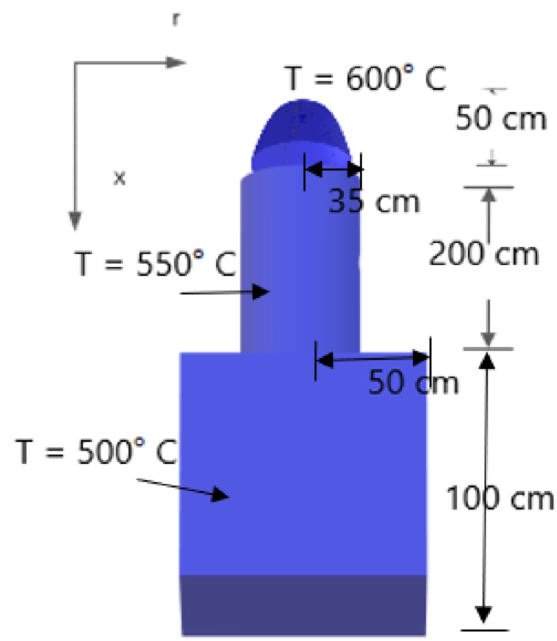


Fig. 6: Shows a schematic summarising the set up of the complex problem

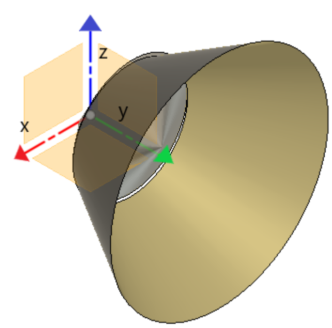


Fig. 7: Shows the tip of the Stirling engine

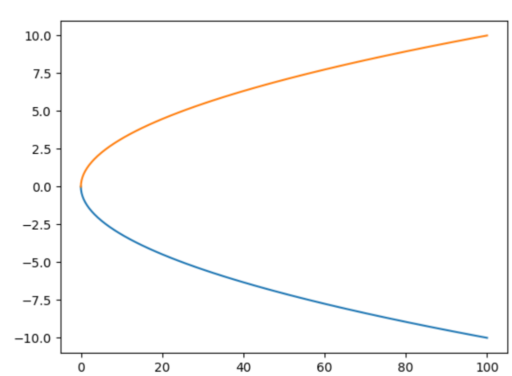


Fig. 8: Shows the cross-sectional area of the tip

TABLE 1: shows values for thermal conductivity and thickness used for the core problem

Material	Thermal Conductivity (W/mK)	Thickness (cm)
Macor	1.45 [16]	2
Aerogel	0.023 [13]	1.5
Tungsten	137 [3]	5

where α can be used to adjust the width of the parabola to make it align with the diameter of the cylinder. Nevertheless, to take the 2d model into 3d and derive a formula for the surface area we can use the surface of revolution equation. by applying the revolution function to transform it to a surface area we get equation 34

$$\int (2\pi x^2 \sqrt{1+(2x)^2} dx \tag{34}$$

To derive an analytical equation for the temperature profile

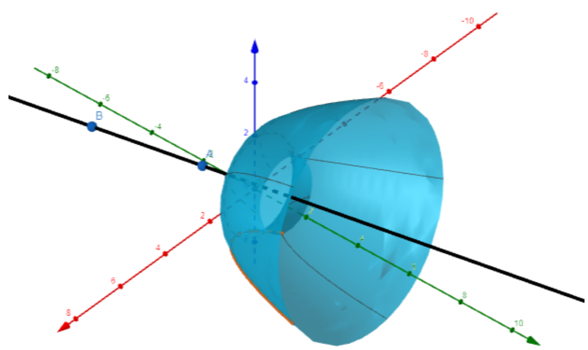


Fig. 9: Shows the cross-sectional area of the tip

we will use a numerical approximation of the value of $\sqrt{1+(2x)^2}$ to solve the integral more easily. Therefore, given that the value of x is always smaller than one:

$$\sqrt{1+4x^2} \approx 1 + \frac{1}{2}(4x^2) - \frac{1}{8}(4x^2)^2 + \frac{1}{16}(4x^2)^3 \tag{35}$$

this can be used to simplify the integral to the following (using only the first three terms of the expansion)

$$2\pi \int x^2 *(1+2x^2-8x^4+4x^6)dx \approx 2\pi(\frac{4}{9}x^9 - \frac{8}{7}x^7 + \frac{2}{5}x^5 + \frac{x^3}{3}) \tag{36}$$

5.2 Assumptions

1) assuming that area is a function of x we can assume that the radial heating is even at the tip and there is no temperature gradient therefore we can assume that Qr = 0
2) we can assume that there is no net heat generation
3) we can assume that $|x| < 1$m this is because although the derivation is made in centimetres the equation will be reported in meters so $x_max = 0.5$.
4) assuming steady state
5) assuming that x is never equal to 0 because x is the distance from the focal point, and we can assume that x is never precisely at the focal point as this moves over time and the physical distance between x and the focal point can only be approximated to be 0 metres.
6) assume that the y and the z axes of the slab (bottom section) has no temperature difference as it is insulated.

5.3 Deriving temperature profile

we can derive the temperature profile over the entire geometry shown in figure **??** by considering each section individually and add all contribution in case of continuity.
The tip After applying the shell balance on the differential

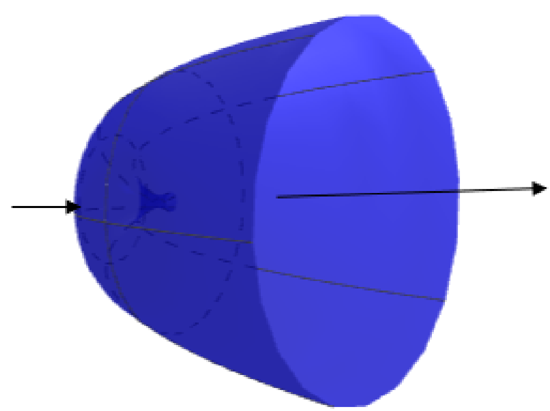


Fig. 10: shows the shell balance on the tip

geometry shown in Figure **??** and substituting the surface area equation (see modelling the geometry section) into the governing equation we get:

$$@q_{x,in} \rightarrow q_x|_x[2\pi(\frac{4}{9}x^9 - \frac{8}{7}x^7 + \frac{2}{5}x^5 + \frac{x^3}{3})] \tag{37}$$

$$@q_{x+\Delta x,out} \rightarrow -q_{x+\Delta x}|_{x+\Delta x}[2\pi(\frac{4}{9}(x+\Delta x)^9 - \frac{8}{7}(x+\Delta x)^7 + \frac{2}{5}(x+\Delta x)^5 + \frac{(x+\Delta x)^3}{3}] \tag{38}$$

After algebraic manipulation we can obtain the following equation:

$$k\frac{d^2T}{dx^2}(\frac{4}{9}(x)^{11} - \frac{8}{7}(x)^9 + \frac{2}{5}(x)^7 + \frac{(5x)^4}{3}) + k\frac{dT}{dx}(\frac{44}{9}x^{10} - \frac{72}{7}(x)^8 + \frac{14}{5}(x)^6 + \frac{(5x)^4}{3}) = 0 \tag{39}$$

let:

$$f(x) = \frac{4}{9}(x)^{11} - \frac{8}{7}(x)^9 + \frac{2}{5}x)^7 + \frac{(x)^5}{3} \tag{40}$$

and

$$g(x) = \frac{44}{9}(x)^{10} - \frac{72}{7}(x)^8 + \frac{14}{5}(x)^6 + \frac{(5x)^4}{3} \tag{41}$$

such that:

$$\frac{d^2T}{dx^2}f(x) + g(x)\frac{dT}{dx} = 0 \tag{42}$$

this is a second order homogeneous ODE that can be solved via substitution. Dividing through by f(x) , this is possible as we are assuming that x is never = 0, we get the following.

$$\frac{d^2T}{dx^2} + \frac{g(x)}{f(x)}\frac{dT}{dx} = 0 \tag{43}$$

This can be solved by substitution to get equation 44

$$T(x) = C_1 + C_2 e^{-\frac{g(x)}{f(x)}x} \tag{44}$$

let u(x)= $-\frac{g(x)}{f(x)}x$ after applying boundary conditions we get:

$$T(x) = 599.9 – 0.0959e^{u(x)} \tag{45}$$

The rod To model the second section, we can use the equation derived earlier:

$$T(x,r) = T_{min} + \Delta T \sum_{n=0}^{\infty} B(sin(\frac{-(J_{mn})^2 x}{L})J_0(j_{0,n}\frac{r}{r_{max}}) \tag{46}$$

The slab the slab can be modelled by the following equation:

$$\nabla^2 T = \frac{\partial^2 T_x}{\partial x^2} + \frac{\partial^2 T_y}{\partial y^2} + \frac{\partial^2 T_z}{\partial z^2} \tag{47}$$

since the sides of the slab are insulated, we can assume that there is no temperature gradient in the y and z direction, therefore our governing equation becomes:

$$\nabla^2 T = \frac{\partial^2 T_x}{\partial x^2} \tag{48}$$

equation 48 is a PDE that can be solved via direct integration to obtain equation 49:

$$T_x = C_1 x + C_2 \tag{49}$$

Applying the boundary conditions we can obtain the following equation:

$$T_x = 675 - 50x \tag{50}$$

Since the solutions are somewhat discontinuous, we can try to model the temperature profile as follows:

$$T(x,r) = \begin{cases} T(x) = 599.9 – 0.0959e^{u(x)} \\ T(x,r) = T_{min} + \Delta T(B(sin(\beta x)J_0(j_{0,1}\frac{r}{r_{max}})) \\ 675 – 50x \end{cases} \tag{51}$$

where $\beta = \frac{-(J_{0,1})^2}{L}$ for the complex problem we can use the same procedure to calculate the cold temperature as done in the core problem. For the slab we can use the following thermal resistance:

$$R_{slab} = \sum(\frac{\Delta x}{kA})_i \tag{52}$$

Then

$$q_x = -137(-0.0959e^{u(0.5)} - 0) = 6863.9\frac{W}{m^2} \tag{53}$$

converting into heatflow

$$\dot{Q}_x = Aq_x = w_{slab}L_{slab}q_x = 0.5 * 1 * 6863.9 = 3431.95W \tag{54}$$

Using the equation for the thermal resistance of a slab we get that the thermal resistance is $1.33\frac{K}{W}$ And using the equation for the percentage change we get a value of 823%.

6 Graphical Solution

For our core problem the temperature profile resulted to be an infinite sum. Therefore, we can sum some terms of the series to investigate whether we can obtain a converging behaviour: As we can see in Figure **??** the temperature does

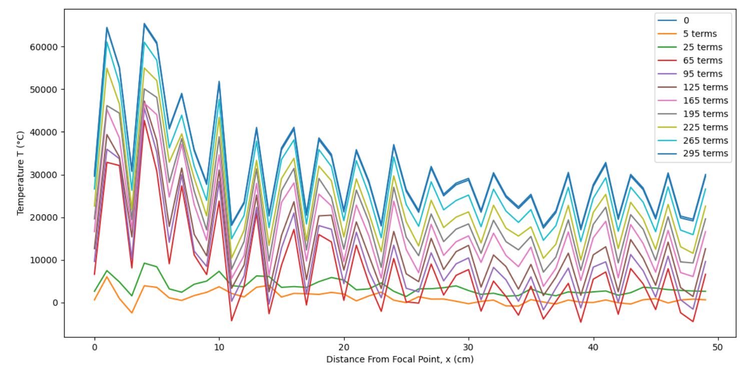


Fig. 11: shows the temperature profile of the core problem after adding n terms to the sum (n is indicated on the legend of the graph

not seem to converge as the spikes get more severe as we increase the number of terms. Nevertheless, the temperature profile varies both in x and r therefore, we can plot the temperature against r for different slices of x as shown in figure **??**. Figure **??** seems to suggest that as the distance from the focal point increases the starting temperature is higher, This does not correspond to what we expected, as such by plotting a 3D graph we might get a more complete picture of how the temperature changes see Figure **??**.

The temperature seems to have a sinusoidal behaviour in x and a decaying temperature in r which is what we expected. To get a more clear picture of the profile of the rod we can plot a heatmap as shown in Figure **??**: The temperature near x = 0 which corresponds to the temperature at the focal point is colder then the temperature further away from the focal point, This discrepancy may be due to the fact that we only used 1 term of the series. by summing 100 terms we get the heatmap illustrated in Figure **??**

This still shows a-physical behaviour, by summing 300 terms we observe that the temperature profile does not converge as Figure **??** suggests. We can compare the temperature profiles of the two models to see whether there are any similarities: The differences in the two seem to be quite severe this may be due to the limitations to our study.

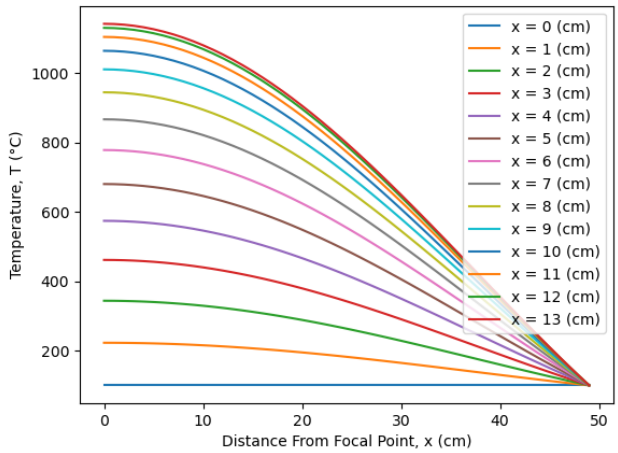


Fig. 12: shows the temperature profile of the core problem for different values of x

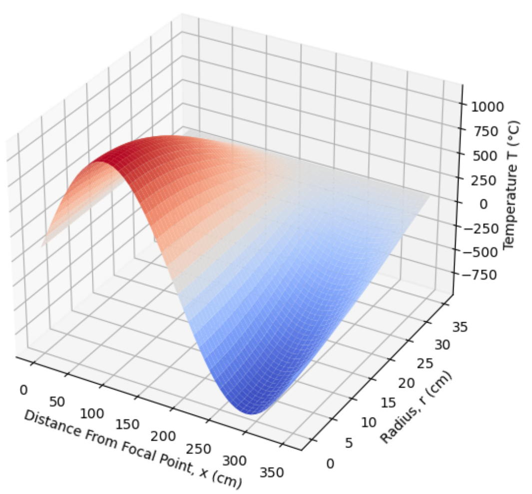


Fig. 13: shows the 3d graph of the temperature profile of the core problem

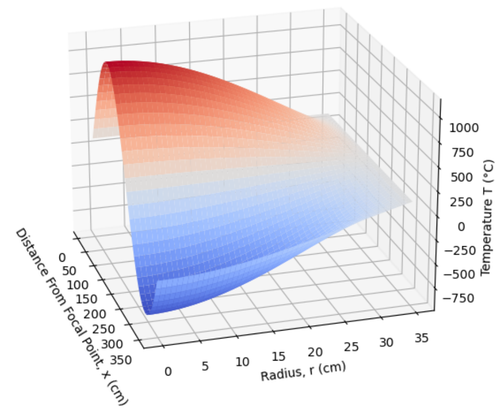


Fig. 14: shows how the 3d graph of the temperature profile of the core problem varies in the $\hat{r}$ direction

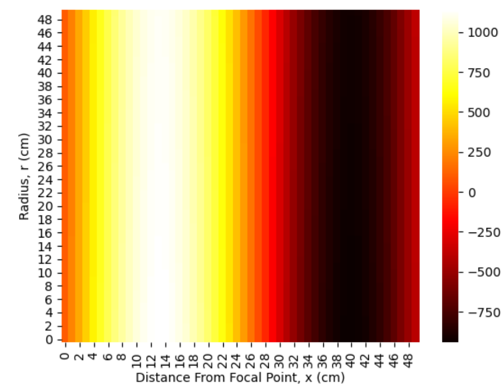


Fig. 15: shows the heatmap of the temperature profile of the core problem with one term of the series

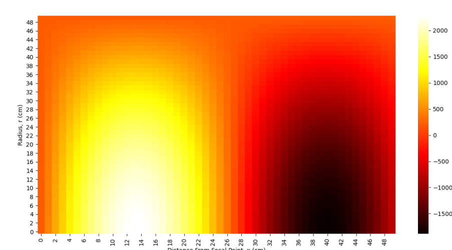


Fig. 16: shows the heatmap of the temperature profile of the core problem after adding 100 terms

6.1 Evaluation

The limitations to our comparison come from the various assumptions that we made throughout the portfolio as well as whether the geometry of the real engine was accurate enough to model a realistic behaviour. As such we can test some of our assumptions to evaluate whether they are adequate for our specific problem. The assumption that can have the largest impact on our results is that of constant thermal conductivity. The assumption that tungsten has constant thermal conductivity was tested by regressing data from [3] plotted in Figure **??** The data was plotted over a large temperature range to observe global behaviour as well as how the thermal conductivity varies around our specific temperature range. Using an exponential regression, the data could be modelled by equation **??**

$$k(T) = 215.89e^{-0.0005T} \tag{55}$$

This would make our integral to get the temperature profile from the heat equation vary. Therefore, this might not be an adequate assumption for large temperature changes below 1000 degrees as the exponential decays rapidly. Another assumption made was that the tip of the cylinder would be placed in proximity to the focal point without considering the thermal resistance of the material. The resistance through the tip of the Stirling engine can be derived from Fourier's law:

$$\dot{Q}_{x,tip} = -kx^2\frac{dT}{dx} \tag{56}$$

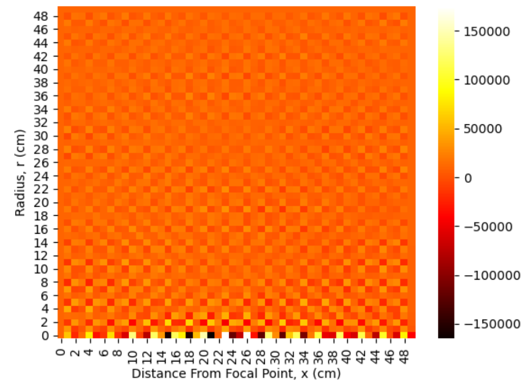


Fig. 17: shows the heatmap of the temperature profile of the core problem after adding 300 terms

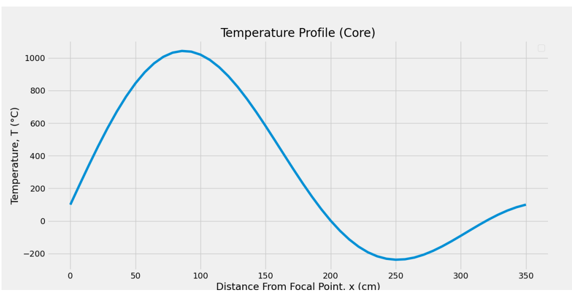


Fig. 18: shows the temperature profile of the core problem after summing one term $@r_{max}$

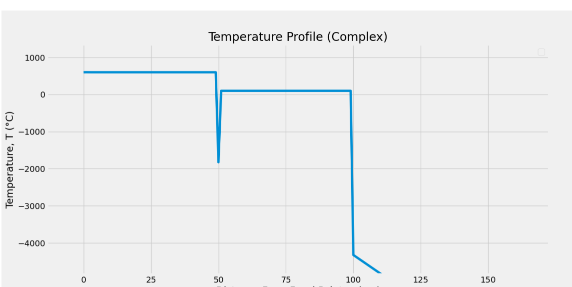


Fig. 19: shows the temperature profile of the complex problem @ r_{max}

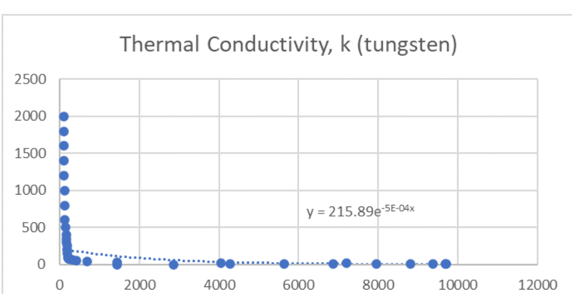


Fig. 20: shows the thermal conductivity of tungsten over a large ΔT

we know that heat flow can be expressed as the ratio between the driving force (temperature difference) over the thermal resistance consequently, we can calculate the thermal resistance by integrating Fourier's Law to get the ΔT term:

$$\int_{x_{bottom}}^{x_{top}} \dot{Q}_{x,tip} x^2 dx = \int_{T_1}^{T_2} -kdT \tag{57}$$

assuming that thermal conductivity is constant over the temperature range integrated we get:

$$-\dot{Q}_{x,tip} \Delta x^{-1} = -k(T_2 - T_1) \tag{58}$$

Therefore:

$$R_{tip} = [k\Delta x]^{-1} \tag{59}$$

calculating this knowing that the tip is 50 cm long and is made of tungsten we get a value for the thermal resistance of approximately 0.0146 $\frac{K}{W}$ this is a small value as such the assumption can be deemed inadequate, which explains why the part of the Temperature profile illustrated in Figure **??** of the previous section that goes from x=0 to x=50 is flat. Finally, the discrepancy in our results could arise simply by the problem being ill defined. This is because most of the boundary conditions as well as the measurements were set arbitrarily at the start to facilitate calculations. Nevertheless, by conducting a thermal analysis on the model using the given boundary conditions we get Figure **??** The

Fig. 21: shows the result of a thermal analysis study conducted in Fusion 360

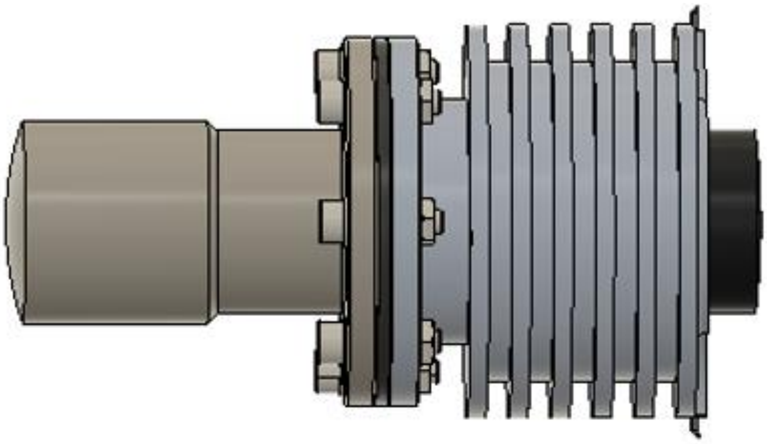
temperature distribution does not appear to reflect what we expected as the tip and the bottom of the large cylinder are at a lower temperature. Therefore, the boundary conditions implemented and/or the measurements might have been inadequate for our problem.

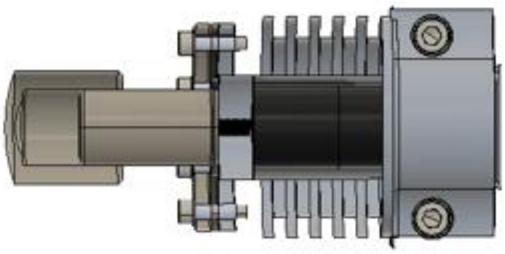
7 Conclusion

In conclusion, the % change in temperature gradient of the aerogel insulated Stirling engine was -29% for the core problem where we used two rods to model the geometry of the engine and 812% for the complex problem where an attempt to model the geometry of the system more rigorously was made. The discrepancy within our result make our evaluation inconclusive, Nevertheless, to improve the study a revision of the model considering the limitations discussed above must be carried out to asses the feasibility of the technology.

References

Appendix





Introduction

The 2nd law of thermodynamics puts a constrain to the thermal efficiency of engines that convert heat into shaft work. This constrain can be expressed by the equation below. As such we can calculate how does an aerogel insulated Stirling engine compare to a traditional one in terms of temperature difference and how their thermal efficiencies compare accordingly. To make this comparison we can assume that thermal efficiency is proportional to the temperature difference between the heat sink and the heat source.

$$\eta \leq 1 - \frac{T_c}{T_h}$$

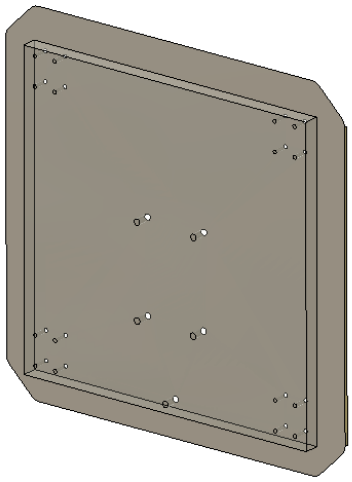
as such if T_c → 0 then eta will go to 1.

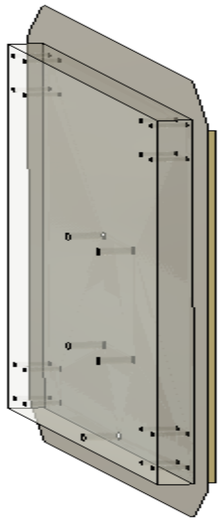
The temperature difference measured using commercial Stirling engines can reach 500 $°C$. Therefore, to calculate the % change of useful heat transferred to the system between the new and the traditional Stirling engine we can use the following:

$$\%\ change = 100 * \left(\frac{new}{traditional} - 1\right)$$

Context

Aerogel is the worlds lightest solid which is 99.8% air and weights only 1.5kg per cubic meter. The material was discovered by a chemical engineer named Samuel Kistler in 1931 and it is currently used by NASA to insulate their materials and manufactured by Aerogel Technology. The insulator can be made by extracting liquids from jellies thereby only leaving the solid structure initially permeated in water, this can be carried out by washing the jelly with a liquid i.e. alcohol and placing the washed jelly on an autoclave where it can reach the critical point to transform the liquid within it into a supercritical fluid and when depressurise the vessel so that where there was liquid before there will be gas. [6]



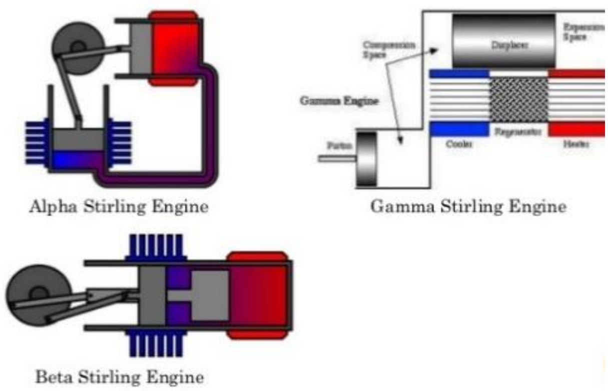


In 1816, Scottish engineer Robert Stirling came up with a way to harness a thermodynamic cycle known to these days as Stirling cycle that consists of using the expansion and contraction of a working fluid to move a piston that would turn a shaft. To convert the mentioned thermodynamic cycle to useful shaft-work the Scottish engineer utilised a buffer space, used to keep the pressure on the right side of the piston constant, he then implemented two pistons, the displacer piston to push the working fluid through the regenerator tubes and a power piston that would move a shaft, more often using a rhombic drive. Additionally, he implemented a chamber inside the regenerator tubes called regenerator which is filled with a porous metal that acts like a sponge that absorbs the heat from the hot fluid and provides the absorbed heat to the cold fluid when it is displaced back to the head of the piston.

Configuration

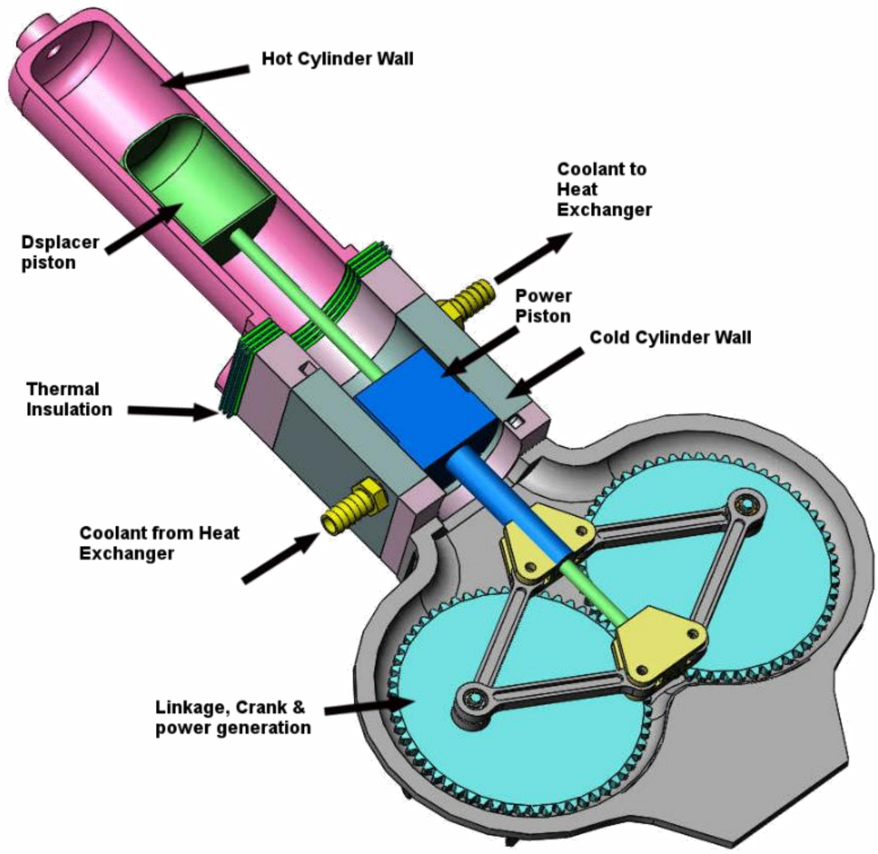
The invention was conceptualised to rival the steam engines that where prominent at the time however they had many problems. Consequently, different configurations of the Stirling engine were developed that would exploit the fluid expansion from the temperature difference in different ways. The most common configurations are:

1. gamma
2. beta
3. alpha



Within this report we will investigate the beta configuration as opposed to the gamma configuration which is often the preferred choice [3] because its configuration makes it easier to insulate with aerogel and make thermal analysis. A schematic of this configuration can be seen below.

[10]



The working fluid that will be considered is hydrogen as this is the best working fluid to use [3]. This is because the Stirling engine works according to Charles law that describes how pressure increases when a fluid is heated at constant volume and decreases when the fluid is cooled, since this principle works best with ideal gasses, we can assess how ideal a gas behave using the compressibility factor which for hydrogen is approximately 1 [4].

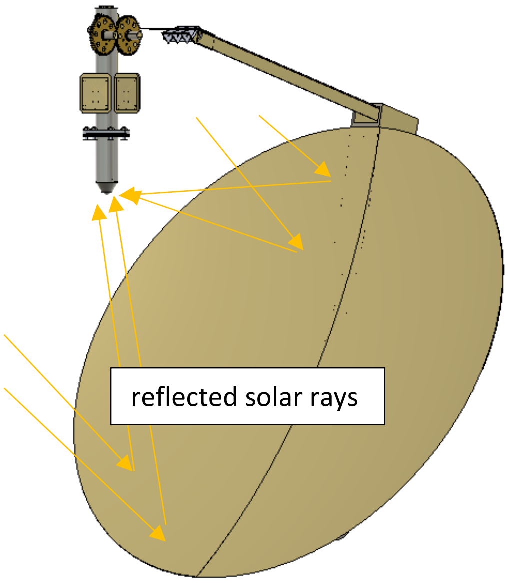
To calculate the heat flow on both configurations we will derive a general model for the temperature profile that would describe both scenarios and then solve for the heat flow using Fourier's law:

$$\dot{Q} = -kA\nabla T$$

by differentiating the temperature profile by applying the nabla operator.

The Model

The Stirling engine that will be modelled can be configured by the following set up



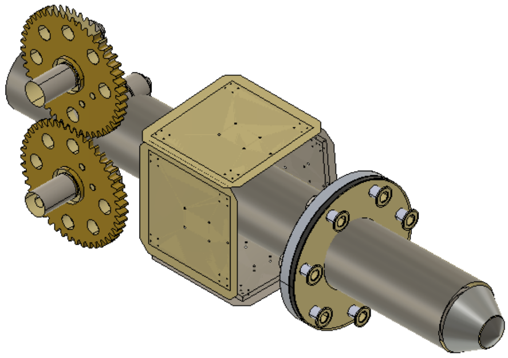
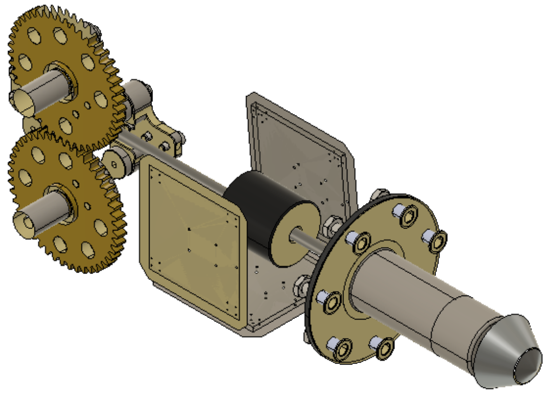
The picture shows a solar parabolic heating dish used to concentrate solar rays to a focal point as shown below

The picture shows a Scheffler reflector which can have a focal point reaching temperatures of 450-650 ° C. [5]. This can be used as the heat source of our problem.

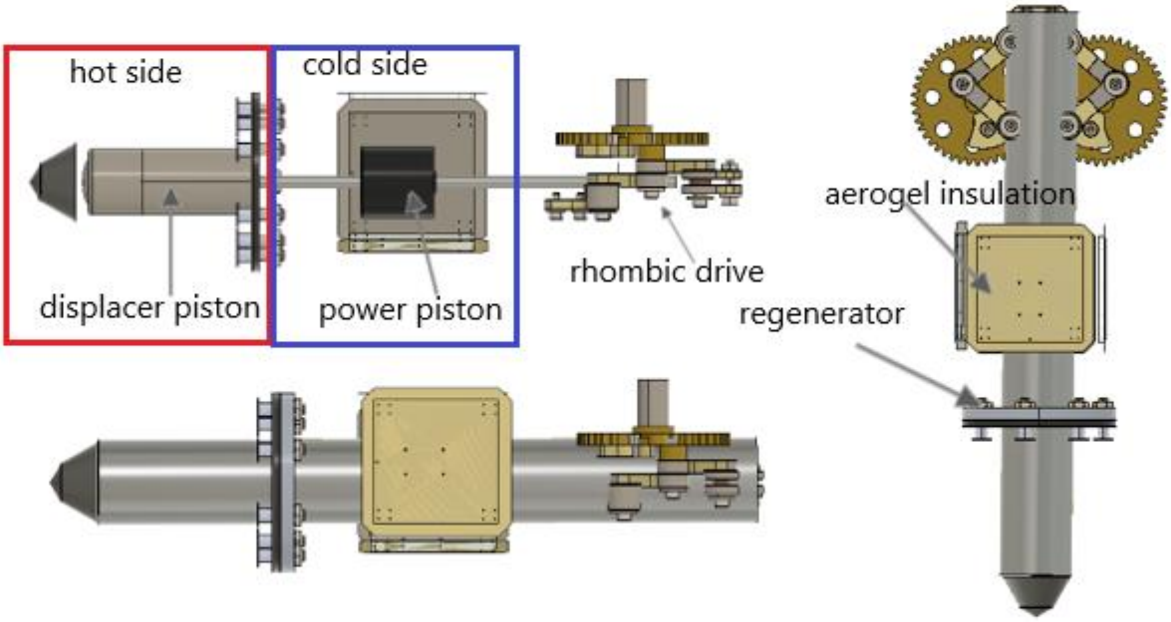
The core problem

Simplifying the Geometry

The first step in solving the core problem consists of simplifying the geometry. A model of the insulated Stirling engine was made using Fusion 360.

Nevertheless, by considering only the sites that are relevant to the heat conduction that needs to be investigated we can divide the engine in two, a hot side (the head of the piston) and a cold side (the insulated side).



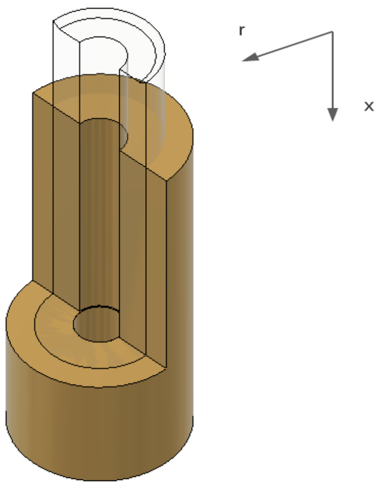
The geometry for the core problem will be simplified to a cylinder or a rod, and since the cold side is wider than the hot side it will be modelled as a larger cylinder. Therefore, the small cylinder represents the hot side closer to the focal point since that is the uninsulated part. As such the problem can be solved by deriving a temperature profile for the entire cylinder (small and large cylinders) to then apply Fourier's law to our model and since the two will be modelled to be two separate objects in contact, if there is continuity between the two solutions, we will write the temperature as a continuous function of x and r. However, if the solution is discontinuous around the interphase between the two cylinders, we will solve for the temperature profile of the large cylinder at

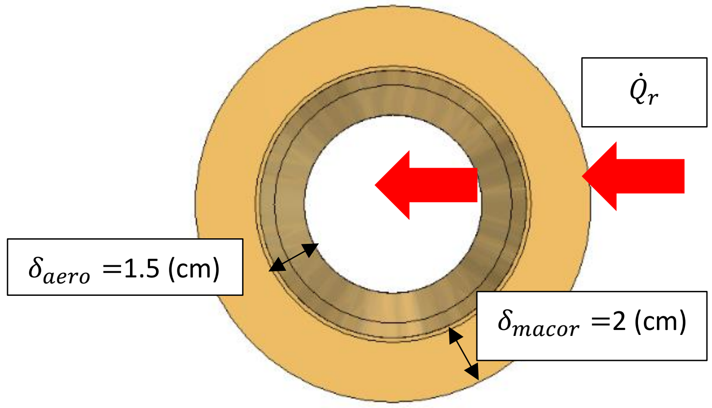
{x: x ∈ $(L, x_{contact})$} and the temperature profile of the smaller cylinder at {x: x ∈ $(x_{contact}, 0)$} and we will express the temperature profile as the following:

$$T(x,r) = \begin{cases} x \in (L, x_{contact}), r : f(x,r) \\ x \in (x_{contact}, 0), r : g(x,r) \end{cases}$$

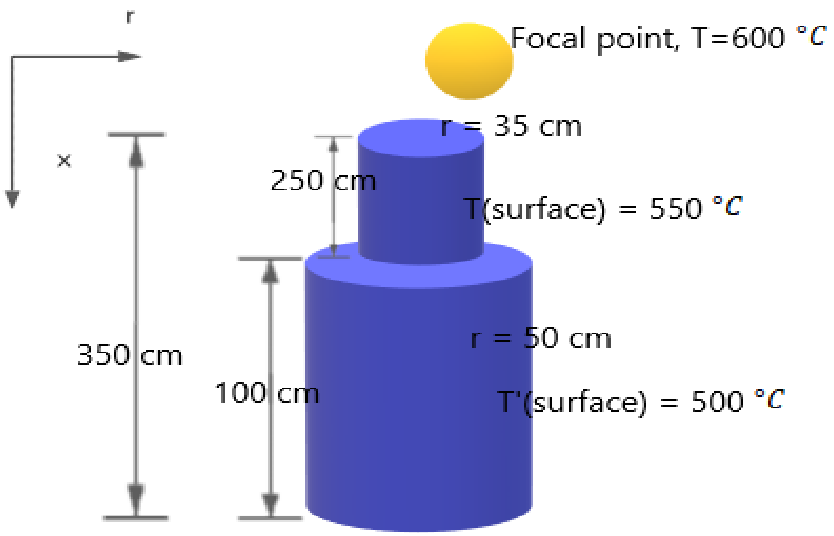
where $x_{contact}$ refers to the height at which the two cylinders are in contact and L is the total height of both cylinders.

The larger cylinder will be modelled as an insulated cylinder in which the x and r coordinates are set as illustrated in the picture below, the heat will be negative as the rod is being radially heated from the -r direction. The x coordinate instead is set to point downward as it will represent the distance from the focal point and since the focal point is close to the smaller cylinder this will increase as we move from the smaller to the larger cylinder, regardless of whether we place the rod upside down or on its side.





The small cylinder instead will be of a different material in this case tungsten and uninsulated. To solve this problem, we will consider the case where the small cylinder is inside the focal point which is the source of heat, and we will assume that the Stirling engine can be sized as follows with the Dirichlet boundary conditions on the two rods set arbitrarily assuming that the average temperature of the smaller cylinders surface is higher as it is closer to the focal point:



Assumptions

1. constant conductivity

Thermal conductivity depends on the material being used. This may be a function of temperature as such it may affect our integral when solving the heat equation. The three materials used for this model are:

- Tungsten in the hot side, this is because has the lowest coefficient of thermal expansion and it heats up very quickly and has high melting point. We can assume that the material has a constant thermal conductivity at high temperatures as shown in [2] and [3].
- Macor is the trademark for a machinable glass-ceramic developed and sold by Corning Inc [17]. The material will be used as insulation and is what the Aerogel is mounted on. This can be considered to have a constant thermal conductivity [16]
- Aerogel also has a constant thermal conductivity, this was reported to be 0.031 [12] at 900 degrees, and 0.01 [13] at 25 degrees. Nevertheless, regardless of the 300% increase the change is only of 0.021 k/W over the temperature range which is small.

2. steady state

Solar irradiance tends to stay constant in the hourly time scale for the bulk of the day [21]. Nevertheless, since our process focuses on the time scale of seconds, we can consider the local irradiation to be constant.

3. only conduction is the mode of heat transfer

For our model we are considering the hot side to be inside the focal point, thereby reducing the effect of air insulation or different forms of convection.

4. Heat does not move in the θ direction

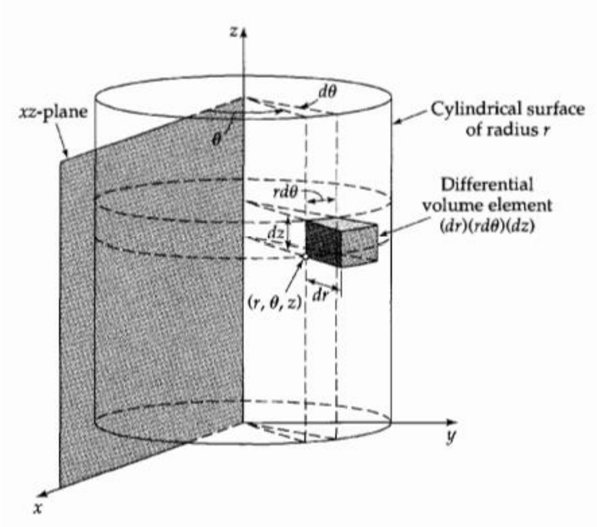
Assuming that heat flows into the rod radially in an homogeneous way since the focal point is modelled as a sphere all the angles will be heated equally as such eliminating a temperature gradient.

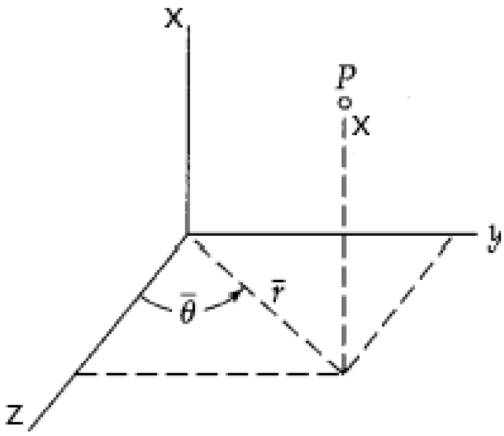
5. No heat generation

since there are no reactions or sources of heat internal to the Stirling engine as the polytropic processes occur adiabatically within the hot and cold section we can assume there is no heat generation.

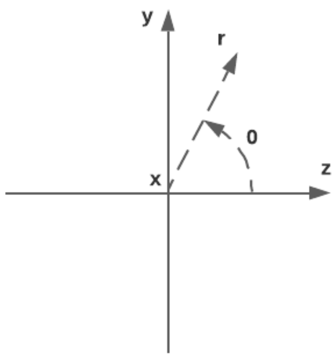
Deriving the temperature profile

to derive the temperature profile, we can apply a heat balance on a differential element. For this problem we will consider the cylindrical coordinates, nevertheless as we consider x to be the distance from the focal point our curvilinear coordinate will be represented by the following:

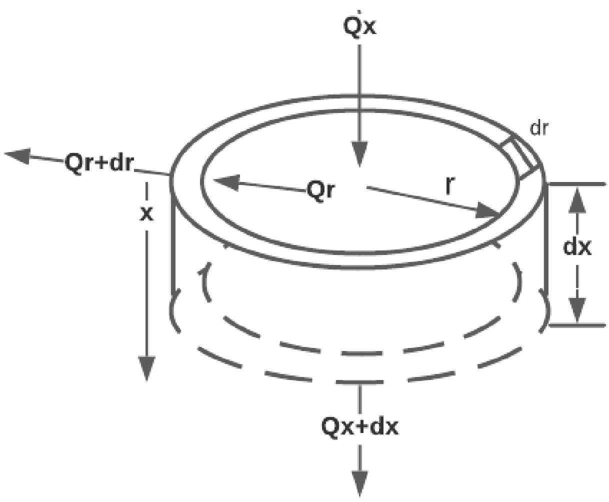




However, since we will not consider the z and the y plane and the x in this case is pointing downwards, we can represent the coordinate system as follows where the x coordinate is pointing downwards with respect to the paper:



The heat flowing in and out of the differential element can be represented by the following as we will not consider the heat moving in the theta direction as mentioned in the assumption section:



The following heat balances can be applied by converting the flow into flux:

1. the heat flowing in through the differential element at x:

$$\dot{Q}_x = q_x A = q_x(\pi(r + \Delta r)^2 - \pi r^2)$$

as the area that is pointing towards is constant and is the difference between the area of the inner and the outer radius of the differential element

2. the heat flowing out of the differential element at x+δx:

$$\dot{Q}_{x+\delta x} = q_{x+\delta x} A = q_{x+\delta x}(\pi(r+\Delta r)^2 - \pi r^2)$$

3. the heat flowing in through the differential element at r:

$$\dot{Q}_r = q_r A = q_r 2\pi r \Delta x$$

the heat flowing out of the differential element at r+δr:

$$\dot{Q}_{r+\delta r} = q_{r+\delta r} A = q_{r+\delta r} 2\pi (r+\Delta r)^2 \Delta x$$

applying the balance by equating heat flux that flows in to the one that flows out as we consider no heat generation at steady state we get:

$$(q_x(\pi(r+\Delta r)^2 - \pi r^2) + q_r 2\pi r \Delta x)_{in} = (q_{x+\delta x}(\pi(r+\Delta r)^2 - \pi r^2) + q_{r+\delta r} 2\pi (r+\Delta r)^2 \Delta x)_{out}$$

$$q_x(\pi(r+\Delta r)^2 - \pi r^2) + q_r 2\pi r \Delta x - q_{x+\delta x}(\pi(r+\Delta r)^2 - \pi r^2) - q_{r+\delta r} 2\pi (r+\Delta r)^2 \Delta x = 0$$

by taking common factors out we get:

$$2\pi(q_x - q_{x+\delta x})(\pi(r+\Delta r)^2 - \pi r^2) + 2\pi \Delta x (q_r r - q_{r+\delta r}(r+\Delta r)^2) = 0$$

dividing through by $2\pi \Delta x \Delta r$ we get:

$$\frac{(q_x - q_{x+\delta x})}{\Delta x} \frac{(\pi(r+\Delta r)^2 - \pi r^2)}{\Delta r} + \frac{q_r r - q_{r+\delta r}(r+\Delta r)^2}{\Delta r} = 0$$

taking the limit as $\Delta r, \Delta x \to 0$

$$\lim_{\Delta r, \Delta x \to 0} \left(\frac{(q_x - q_{x+\delta x})}{\Delta x} \frac{(\pi(r+\Delta r)^2 - \pi r^2)}{\Delta r} + \frac{q_r r - q_{r+\delta r}(r+\Delta r)^2}{\Delta r} \right) = 0$$

$$\frac{\partial q_x}{\partial x} r^2 + r \frac{\partial}{\partial r}(r q_r) = 0$$

taking r squared as a common factor, we will ignore the trivial solutions:

$$r^2 \left(\frac{\partial q_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r q_r) \right) = 0$$

$$r = r_1 = r_2 = 0$$

$$\frac{\partial q_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r q_r) = 0$$

replacing the following with Fourier's law to express the pde in term of temperature:

$$\frac{\partial}{\partial x}(k \frac{dT}{dx}) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk \frac{dT}{dr} \right) = 0$$

we can take the constant out of the partial derivative and divide through by k

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) = 0$$

this is Laplace's equation in cylindrical coordinate for $\frac{dT}{d\theta} = 0$

To solve the derived second order homogeneous partial differential equation, we will make our variables dimensionless and use the following boundary conditions:

1. $Dirichlet\ B.C. \rightarrow T(x = 0, r) = T_{min};$
2. $Dirichlet\ B.C. \rightarrow T\left(x = L, r = \frac{r_{max}}{2}\right) = T_{max};$

the following non-dimensional parameters will be used, xi will be the negative distance from the focal point as such it will be the length of the Stirling engine :

$$\theta = \frac{T - T_{min}}{T_{max} - T_{min}};$$

$$\rho = \frac{r}{r_{max}};$$

$$\xi = -\frac{x}{L};$$

in differential form we get:

$$\partial\theta = \frac{1}{T_{max} - T_{min}}\partial T;$$

$$\partial\rho = \frac{1}{r_{max}}\partial r;$$

$$\partial\xi = -\frac{1}{L}\partial x;$$

the PDE can be transformed into the following by applying the product rule and can be non-depersonalised:

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x}\left(\frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial \xi}\frac{\partial \xi}{\partial x}\right) = -\frac{T_2 - T_1}{L^2}\frac{\partial^2 \theta}{\partial \xi^2}$$

$$\frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial \rho}{\partial r}\frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial \rho} = \frac{T_2 - T_1}{\rho r_{max}^2}\frac{\partial \theta}{\partial \rho}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right) = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial r}\left(\frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial \rho}\frac{\partial \rho}{\partial x}\right) = \frac{(T_2 - T_1)}{r_{max}^2}\frac{\partial^2 \theta}{\partial \rho^2}$$

This becomes:

$$-\frac{T_2 - T_1}{L^2}\frac{\partial^2 \theta}{\partial \xi^2} + \frac{T_2 - T_1}{\rho r_{max}^2}\frac{\partial \theta}{\partial \rho} + \frac{(T_2 - T_1)}{r_{max}^2}\frac{\partial^2 \theta}{\partial \rho^2} = 0$$

multiplying through by: $\frac{r_{max}^2}{T_2 - T_1};$

we get:

$$-\frac{1}{L^2}\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{\rho}\frac{\partial\theta}{\partial\rho}+\frac{\partial^2\theta}{\partial\rho^2};$$

To solve this equation, we can make the following assumption:

$$\theta(\xi,\rho)=X(\xi)R(\rho)$$

This will make our boundary conditions to be:

1. $T(x=0,r)=T_{min}=\theta(0,\rho)=0=X(0)R(\rho)$
2. $T\left(x=L,r=\frac{r_{max}}{2}\right)=T_{max}=\theta(1{,}0.5)=1=X(1)R(0.5)$

To find these separable functions we can substitute the separable functions into the PDE:

$$-\frac{R}{L^2}\frac{\partial^2 X}{\partial\xi^2}+\frac{X}{\rho}\frac{\partial R}{\partial\rho}+X\frac{\partial^2 R}{\partial\rho^2}=0$$

by grouping the terms we can separate the variables:

$$R\left(-\frac{1}{L^2}\frac{\partial^2 X}{\partial\xi^2}\right)=-X\left(\frac{1}{\rho}\frac{\partial R}{\partial\rho}+\frac{\partial^2 R}{\partial\rho^2}\right)$$

dividing through by RX/L^2 we can see that if xi is changing for a fixed position rho then the RHS of the equation is constant , so the LHS must also be constant, we will set this constant as k^2which will be the eigenvalue of our function.

$$-\frac{1}{X}\left(\frac{\partial^2 X}{\partial\xi^2}\right)=-\frac{L^2}{R}\left(\frac{1}{\rho}\frac{\partial R}{\partial\rho}+\frac{\partial^2 R}{\partial\rho^2}\right)=\pm k^2$$

we will take the case where $k^2$is positive so that our X variable will become a particular PDE that fall into the category of Sturm-Liouville Equations of the form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=k^2r(x)y$$

in our case $p(x)=1, q(x)=0\ and\ r(x)=1.$

$$\frac{\partial^2 X}{\partial\xi^2}+k^2X=0$$

for a positive eigenvalue k squared (that is the only scenario that doesn't lead to a-physical behaviours) we get:

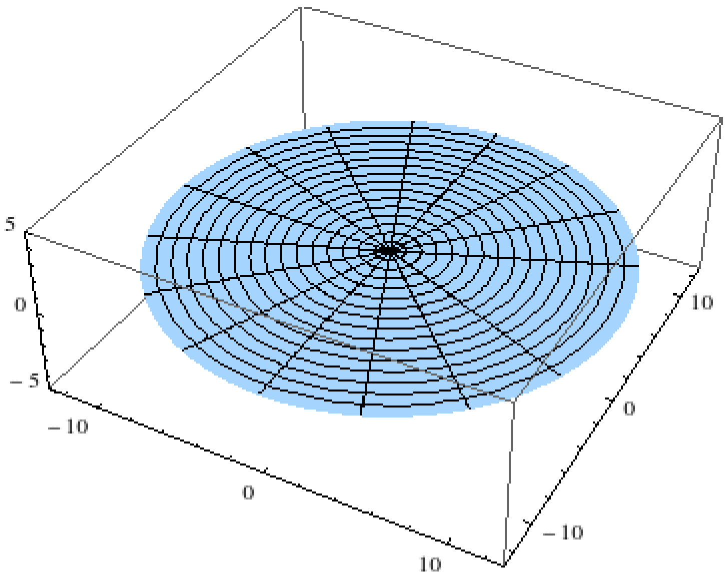
$$X(\xi)=Asin(k^2\xi)+Bcos(k^2\xi)$$

The second equation instead:

$$\frac{1}{R}\left(\frac{1}{\rho}\frac{\partial R}{\partial\rho}+\frac{\partial^2 R}{\partial\rho^2}\right)+\left(\frac{k}{L}\right)^2=0$$

This equation falls into the category of Bessel functions of the form:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-\alpha^2)y=0$$



In our case the value of alpha is zero and k/L squared is 1 therefore we find that $k = L$:

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \frac{\partial R}{\partial \rho} + \rho^2 \left(\frac{k}{L}\right)^2 R = 0$$

The solution to this equation can be found using a Taylor expansion and can be written as :

$$J_\alpha(\lambda x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{\lambda x}{2}\right)^{2m+\alpha}$$

where the gamma function is a shifted generalised factorial that can be expressed as the following integral using Laplace transform:

$$\Gamma(n) = (n-1)! = \int_0^\infty x^{n-1} e^{-x} dx$$

$$R(\rho) = j_m(\lambda\rho)$$

in our case alpha is qeual to 0, therefore:

$$j_0(\lambda x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1)} \left(\frac{\lambda x}{2}\right)^{2m}$$

where lambda is our eigen value.

To find the constants of the function we can apply the first B.C. which constrains the values of our equations to:

$$\theta(0,1) = 0 = X(0)R(1)$$

substituting these values, we get:

$$0 = [Asin(0) + Bcos(0)]j_m(\lambda)$$

This equation is true when:

$$0 = Asin(0) + Bcos(0)$$

or when:

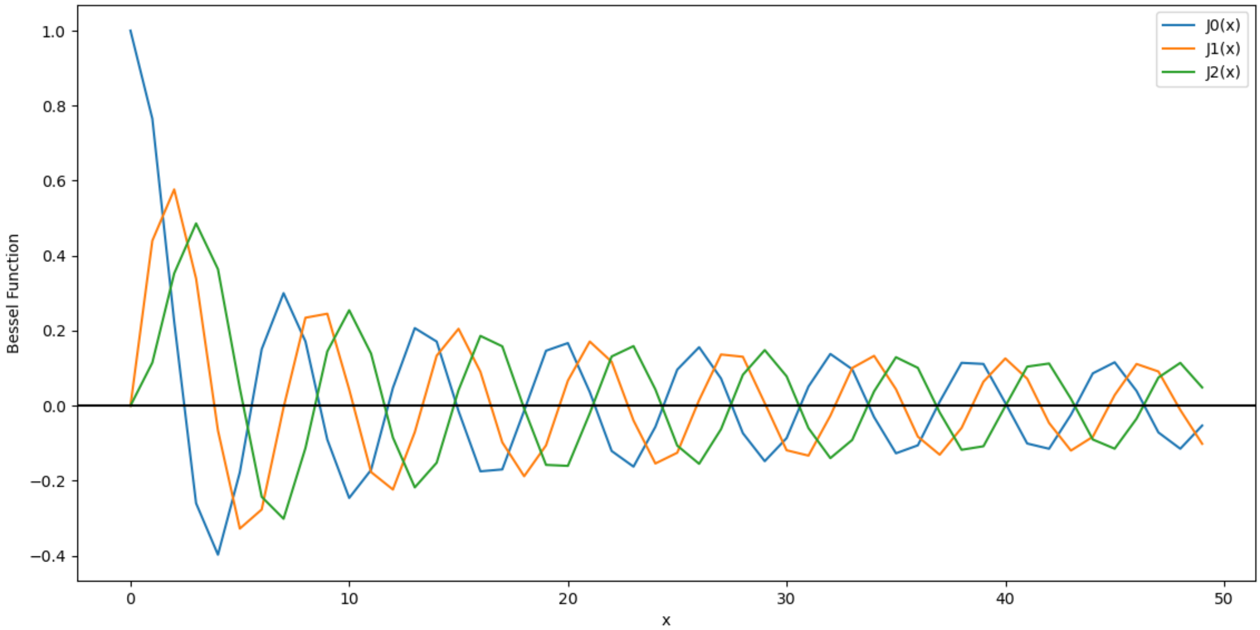
$$0 = j_m(\lambda)$$

and when both above equations are true.

For the first equation to be true we get that:

$$B = 0$$

for the second equation to be true, the constant lambda must be a root of J_m(p) as such one of the eigenvalues of the function. By observing $j_m(p)$ plotted for different values of m, we can see that there are periodic roots that change depending on the value of m.



Let these roots be J such that:

$$J(J_{mn}) = 0$$

These values can be looked up in standard reference books for example , j_01=2.405 and j_02= 5.520, these will be our eigenvalues, therefore, lambda will be j_mn.

Now we have only one constant of integration to solve for:

$$\theta(\rho, \xi) = Asin(\mathrm{j}_{\mathrm{mn}}^2 \xi) j(J_{mn}\rho)$$

applying the second boundary condition we get that:

$$1 = Asin(j_{mn}^2) j\left(\frac{j_{mn}}{2}\right)$$

$$\therefore$$

$$A = \frac{1}{sin(j_{mn}^2) j\left(\frac{j_{mn}}{2}\right)}$$

Now, we can apply the principle of superposition to add up all the instances of m and n as they are all solutions that satisfy our boundary conditions, This makes our general solution to be:

$$\theta(\rho,\xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2)j\left(\frac{j_{mn}}{2}\right)} sin(j_{mn}^2\xi)j_m(j_{mn}\rho)$$

For our case we can only apply this solution for the case of m = 0 therefore:

$$\theta(\rho,\xi) = \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2)j\left(\frac{j_{0,n}}{2}\right)} sin(j_{mn}^2\xi)j_0(j_{0,n}\rho)$$

Therefore, the temperature profile is:

$$\frac{T - T_{min}}{T_{max} - T_{min}} = \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2)j\left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right)j_0(j_{0,n}\frac{r}{r_{max}})$$

$$T(x,r) = T_{min} + (T_{\max} - T_{min})\sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2)j_0\left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right)j_0(j_{0,n}\frac{r}{r_{max}})$$

Evaluation

Now that we have a temperature profile for the entire Stirling engine model, we can calculate the heat that goes into the system by applying the nabla operator and to maximise the amount of useful heat that goes into the system we need to minimise the thermal resistance:

$$Q_{in} = \frac{\Delta T}{R_{tot}} = \frac{T_c - T_h}{R_{tot}}$$

to calculate the thermal resistance in a cylinder we can use the following :

$$R_{tot} = \sum\left(\frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}\right)_i$$

where i refers to each layer of insulation that heat transfers through, to calculate the thermal conductivity we can assume that the layer or Macor machinable glass that the Aerogel is mounted on is merged with the Aerogel such that the thermal conductivity of the insulation layer will be the weighted average of the two.

Materia	Thermal Conductivity $\frac{W}{mK}$	Thickness δ(cm)
Macor	1.45 [16]	2
Aerogel	0.023 [13]	1.5
Tungsten	137 [3]	5

$$R_{tot} = R_{aerogel} + R_{macor} = \sum\left(\frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}\right)_i$$

$$R_{tot} = \frac{\ln\left(\frac{r_{max}}{r_{max} - \delta_{aerogel}}\right)}{2\pi k_{aerogel}(L - x_{contact})} + \frac{\ln\left(\frac{r_{max}}{r_{max} - \delta_{aerogel}}\right)}{2\pi k_{aerogel}(L - x_{contact})}$$

$$R_{tot} = \frac{\ln\left(\frac{0.5}{0.5 - 0.015}\right)}{2\pi 0.023 * 1} + \frac{\ln\left(\frac{0.5}{0.5 - 0.02}\right)}{2\pi * 1.5 * 1} \approx 0.215 \frac{k}{W}$$

To calculate the cold temperature of the system we can use Fourier's law by differentiating the temperature profile. This can be written as follows:

$$q = -k\nabla T$$

where our temperature gradient is:

$$\nabla T = \frac{\partial T}{\partial r}\boldsymbol{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{\theta} + \frac{\partial T}{\partial x}\boldsymbol{x}$$

The thermal conductivity k in this equation is a second order tensor or a 3x3 matrix. To find the entries to the transformation we could use a numerical algorithm as done in [18].

$$k = \begin{pmatrix} k_{rr} & k_{rx} & k_{r\theta} \\ k_{xr} & k_{xx} & k_{x\theta} \\ k_{\theta r} & k_{\theta x} & k_{\theta\theta} \end{pmatrix}$$

Pure tungsten is homogeneous, and its crystalline structure exhibits mainly two forms alpha or beta [19]. Nevertheless, depending on the way it has been manufactured crystallographic defects, such as vacancies, grain boundaries and dislocations may affect the thermal conductivity of tungsten as it could exhibit anisotropic behaviour[20]. However, since tungsten forms very strong metallic bonds, dislocations are unlikely to occur, as such we can assume that the thermal conductivity tensor transforms the r, x and theta components of the T vector equally by scalar k. This corresponds to an isotropic scalar k which can be represented as follows:

$$k = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

this can be simplified as follows:

$$k = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = k * I \approx k$$

Where I represent the identity matrix. The result makes sense since pure metals often have a constant thermal conductivity and are isotropic.

Therefore, we can calculate our temperature gradient:

$$T(x, r) = T_{min} + (T_{\max} - T_{min}) \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2) j_0\left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right) j_0(j_{0,n}\frac{r}{r_{max}})$$

$$\nabla T = \frac{\partial T}{\partial r}\boldsymbol{r} + \frac{\partial T}{\partial x}\boldsymbol{x}$$
$$= \begin{cases} (T_{\max} - T_{min}) \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2) j_0\left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right) \frac{j_{0,n}}{2} \frac{n}{r_{max}} \left(\frac{(-1)^n}{n!\,\Gamma(n+1)}\right) \left(\frac{j_{0,n}}{2}\frac{r}{r_{max}}\right)^{2n-1} & \boldsymbol{r} \\ \frac{j_{mn}^2}{L}(T_{min} - T_{max}) \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2) j_0\left(\frac{j_{0,n}}{2}\right)} cos\left(-\frac{j_{mn}^2 x}{L}\right) j_0(j_{0,n}\frac{r}{r_{max}})\, \boldsymbol{x} & \end{cases}$$

where **r,x** represents the unit vectors in the r and x direction respectively, and the derivative of the Bessel function was computed assuming that the derivative of the sum is the sum of the derivative.

Since we are considering the heat coming from the hot side into the cold side, the temperature gradient will be evaluated at: x=2.5m and r=0.35m. Therefore, we can calculate the **r** and **x** components of the heat flow vector as follows:

$$\dot{Q}_x = -Ak\frac{\partial T}{\partial x} = 2\pi r L_{small\,rod} k_{tungsten}\left(\frac{\partial T}{\partial x}\right)_{x=2.5, r=0.35}$$

$$\dot{Q}_r = -Ak\frac{\partial T}{\partial r} = -2\pi r L_{small\,rod} k_{tungsten}\left(\frac{\partial T}{\partial r}\right)_{x=2.5, r=0.35}$$

As we expected the **x** component of the heat flow vector is positive, this is because we set the x coordinates to go in the same direction of heat. Evaluating the two components we get:

$$\dot{Q}_r = -2711465.42W$$
$$\dot{Q}_x = 3.495 * 10^{-10}\,W$$

to calculate ΔT now we can equate the x component of the heat flow coming from the hot rod to the thermal resistance equation from the cold rod, this is because the heat that is transferred from one rod to the other moves into the cold side axially:

$$\dot{Q} = -Ak\frac{\partial T}{\partial x} = \frac{\Delta T}{R_{tot}} = \frac{T_c - T_h}{R_{tot}} = 3.495 * 10^{-10}W$$

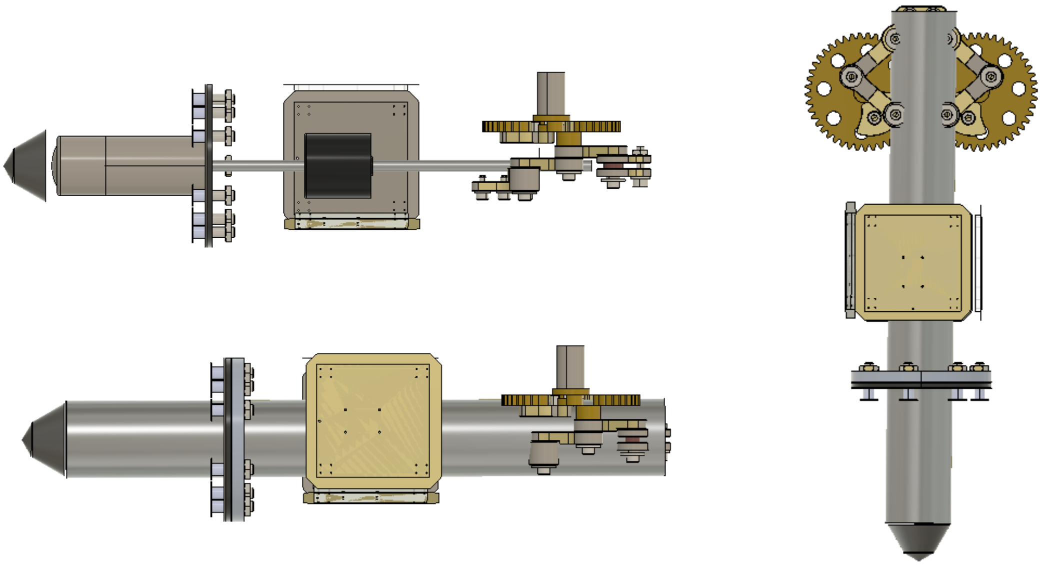
$$\Delta T = 3.495 * 10^{-10} * R_{tot} \approx 0K$$

$$\%change = 100 * \left(\frac{0}{500} - 1\right) = -100\%$$

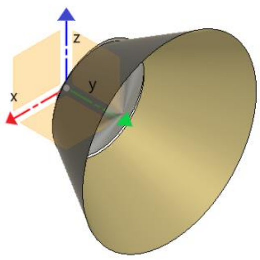
The complex problem

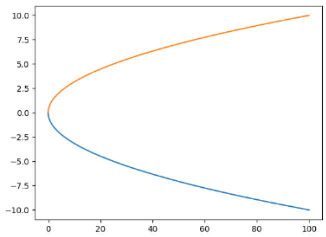
Modelling the geometry

for the complex problem we will consider a geometry of the first cylinder that approximates to the actual Stirling engine a bit more, this can be done by dividing our geometry into two parts, the tip of the cylinder and the body of the cylinder.



The tip of the cylinder can be shown below

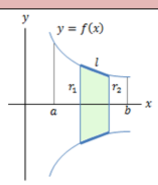




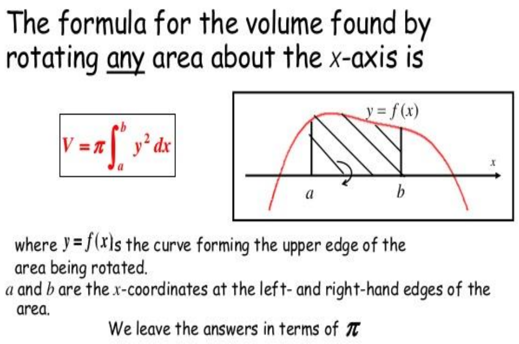
Upon visual inspection we can observe that the area of the tip increases with increasing x (distance from the top). Therefore, we can model the cross-sectional area of the tip as a quadratic plotted on the right. A good approximation to the cross-sectional area can be

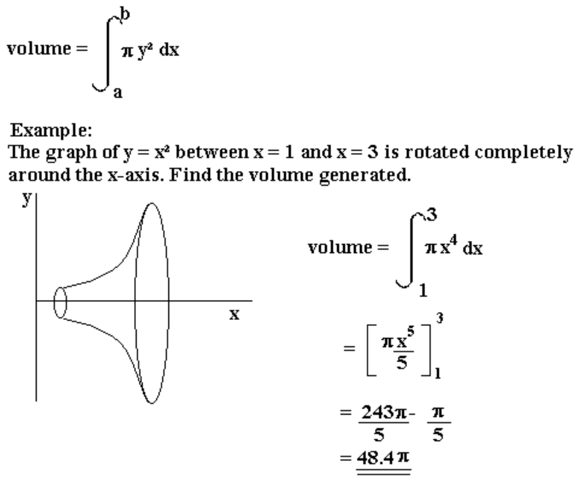
$$A = \alpha x^2$$

where α can be used to adjust the width of the parabola to make it align with the diameter of the cylinder. Nevertheless, to take the 2d model into 3d to get a formula for the surface area we can use the surface of revolution equation. The general function of the equation can be seen below.

Area of a surface of revolution – Summary:	
	*If a curve $f(x)$ is positive in the region a to b then the area of the surface of revolution can be found by dividing the surface to small frastrum cones and summing the area of each cone which is: $s = \pi(r_1 + r_2)l$ were r_1 and r_2 are the two radii of the frastrum cone.*
*The curve $y = f(x)$ is revolved about **x axis**:*	$A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$
*The curve $y = f(x)$ is revolved about **y axis**:*	$A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx$
Arc length:	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

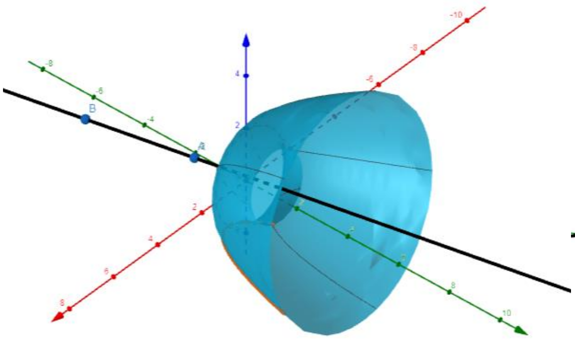
whereas the formula for the volume of a revolution:

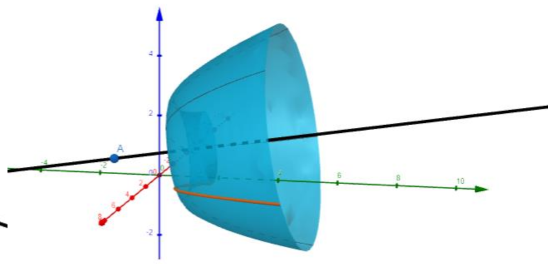




we know that A is a function of A by looking at the tip of the head and a cross-sectional area of the tip.

and by applying the revolution function to transform it to a surface area we get the following:





this can be modelled by the following equation :

$$SA = \sum\left(2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\Delta x\right)$$

knowing that $y = x^2$ and $\frac{dy}{dx} = 2x$

we get the following equation by substituting SA into the governing equation

$$\sum\left(2\pi x^2\sqrt{1+(2x)^2}\Delta x\right)$$

taking the limit as $\Delta x \to 0$

$$\int 2\pi x^2\sqrt{1+(2x)^2}\mathrm{d}x$$

To derive an analytical equation for the temperature profile we will use a numerical approximation of the value of $\sqrt{1+(2x)^2}$ to solve the integral more easily

this is because the integral becomes

$$\int 2\pi x^2\sqrt{1+(2x)^2} = \frac{1}{64}\sqrt{4x^2+1}\,(16x^3+2x) - \frac{1}{64}\ln\left(\left|\sqrt{4x^2+1}+2x\right|\right) + C$$

which makes the governing equation harder to solve.

Therefore, given that the value of x is always smaller than one as state on the assumptions section we can make the following approximation:

$$\sqrt{4x^2+1} \approx 1 + \frac{1}{2}(4x^2) - \frac{1}{8}(4x^2)^2 + \frac{1}{16}(4x^2)^3$$

from the following formula:

(c) $\sqrt{1+x}$ Write in index form

$= (1+x)^{\frac{1}{2}}$ Use expansion with $n = \frac{1}{2}$ and x replaced with x

$$= 1 + \left(\frac{1}{2}\right)(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2\,!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^3}{3\,!} + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

Expansion is infinite. Valid when $|x| < 1$.

this can be used to simplify the integral

$$2\pi\int x^2\sqrt{4x^2+1}\,dx$$

to the following (using only the first three terms of the expansion)

$$2\pi\int x^2 * (1 + 2x^2 - 8x^4 + 4x^6)dx = 2\pi\int x^2 + 2x^4 - 8x^6 + 4x^8 dx$$

that can be solved more easily as:

$$2\pi\int x^2\sqrt{4x^2+1}\,dx \approx 2\pi\left(\frac{4}{9}x^9 - \frac{8}{7}x^7 + \frac{2}{5}x^5 + \frac{x^3}{3}\right)$$

The second section of the temperature profile is a cylinder, and the third section is a slab the following image shows the section of the Stirling engine being modelled. Finally, the size of the different sections with the boundary conditions is illustrated below:

Assumptions

1. assuming that:

$$A = F(x)$$

2. we can assume that the radial heating is even at the tip and there is no temperature gradient therefore we can assume that $Q_r = 0$
3. we can assume that there is no net heat generation
4. we can assume that |x| < 1m this is because although the derivation is made in centimetres the equation will be reported in meters so x_max = 0.5.
5. assuming steady state

6. assuming that x is never equal to 0 because x is the distance from the focal point, and we can assume that x is never precisely at the focal point as this moves over time and the physical distance between x and the focal point can only be approximated to be 0 metres.
7. assume that the y and the z axes of the slab (bottom section) has no temperature difference as it is insulated.

Deriving temperature profile

we can derive the temperature profile from the governing equation

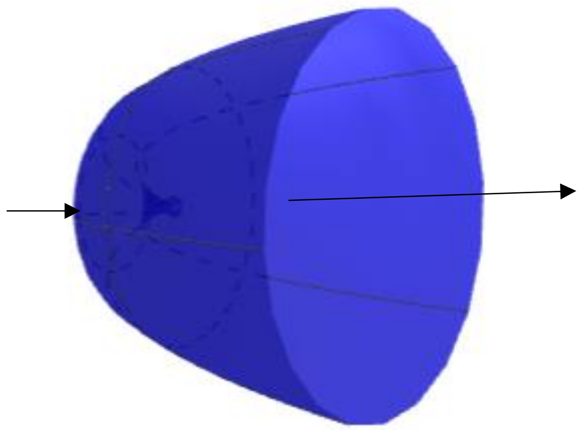
$$accumulation = in - out + gen$$

this is assuming that the accumulation of heat is equal to zero due to steady state flow:

$$0 = in - out + gen$$

for the heat flux we get:

$$0 = q_x|_x A - q_x|_{x+\delta x} A + \dot{q_v} dV$$



substituting the surface area equation (see modelling the geometry section)into the governing equation we get:

$$+q_x|_x \left(2\pi\left(\frac{4}{9}x^9 - \frac{8}{7}x^7 + \frac{2}{5}x^5 + \frac{x^3}{3}\right)\right) @q_x$$

$$-q_x|_{x+\delta x} \left(2\pi\left(\frac{4}{9}x + \Delta x^9 - \frac{8}{7}x + \Delta x^7 + \frac{2}{5}x + \Delta x^5 + \frac{x + \Delta x^3}{3}\right)\right) @q_{x+\delta x}$$

$$+\dot{q_v} dV = 0$$

using the formula from the volume of a revolution, (see modelling the geometry section), we get:

$$dV = \int \pi y^2 dx = \pi \int x^4 dx = \pi \frac{1}{5} x^5$$

therefore, the equation becomes the following as we divide through by 2pi:

$$q_x|_x\left(\frac{4}{9}x^9-\frac{8}{7}x^7+\frac{2}{5}x^5+\frac{x^3}{3}\right)$$
$$-\,q_x|_{x+\delta x}\left(\frac{4}{9}(x+\Delta x)^9-\frac{8}{7}(x+\Delta x)^7+\frac{2}{5}(x+\Delta x)^5+\frac{(x+\Delta x)^3}{3}\right)$$
$$+\dot{q_v}\frac{1}{10}x^5=0$$

if we divide through by delta x and we consider the case for no heat generation, taking the limit as $\Delta x \to 0$

we get

$$-\frac{d}{dx}\left(q_x*\left(\frac{4}{9}x^9-\frac{8}{7}x^7+\frac{2}{5}x^5+\frac{x^3}{3}\right)\right)+\ \dot{q_v}\frac{1}{10}x^5=0$$

replacing the flux with Fourier's law

$$\frac{d}{dx}\left(x^2k\frac{dT}{dx}*\left(\frac{4}{9}x^9-\frac{8}{7}x^7+\frac{2}{5}x^5+\frac{x^3}{3}\right)\right)=0$$

using the product rule after multiplying through the area term (x^2)

$$k\frac{d^2T}{dx^2}\left(\frac{4}{9}x^{11}-\frac{8}{7}x^9+\frac{2}{5}x^7+\frac{x^5}{3}\right)+\ k\frac{dT}{dx}\left(\frac{44}{9}x^{10}-\frac{72}{7}x^8+\frac{14}{5}x^6+\frac{5x^4}{3}\right)=0$$

let:

$$f(x)=\left(\frac{4}{9}x^{11}-\frac{8}{7}x^9+\frac{2}{5}x^7+\frac{x^5}{3}\right)$$
$$g(x)=\left(\frac{44}{9}x^{10}-\frac{72}{7}x^8+\frac{14}{5}x^6+\frac{5x^4}{3}\right)$$

such that:

$$\frac{d^2T}{dx^2}f(x)+g(x)\frac{dT}{dx}=0$$

this is a second order ordinary homogeneous differential equation that can be solved via substitution. Dividing through by $f(x)$, this is possible as we are assuming that x is never = 0, we get the following.

$$\frac{d^2T}{dx^2}+\frac{g(x)}{f(x)}\frac{dT}{dx}=0$$

we know that T is a function of x so let's try a solution of the form:

$$T(x)=e^{\lambda x}$$

then:

$$\frac{dT}{dx}=\lambda e^{\lambda x}$$

$$\frac{d^2T}{dx^2} = \lambda^2 e^{\lambda x}$$

if we substitute the following back into the equation

$$\lambda^2 e^{\lambda x} + \frac{g(x)}{f(x)} \lambda e^{\lambda x} = 0$$

dividing through by $e^{\lambda x}$

we get

$$\lambda^2 + \frac{g(x)}{f(x)} \lambda = 0$$

therefore, the value of lambda must be:

$$\lambda \left(\lambda + \frac{g(x)}{f(x)} \right) = 0$$

$$\therefore$$

$$\lambda = \{0, -\frac{g(x)}{f(x)}\}$$

using superposition, we get that the temperature profile at the tip can be described by the following:

$$T(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$T(x) = \ C_1 e^{0x} + C_2 e^{-\frac{g(x)}{f(x)} x}$$

$$\therefore$$

$$T(x) = C_1 + C_2 e^{-\frac{g(x)}{f(x)} x}$$

the first boundary condition that we can apply is that for small values of x $T(x_{top}) \rightarrow T_{max}$ since it is close to the temperature of the heat source

$$T(x \rightarrow 0) \approx 600$$

let $u(x) = \ -\frac{g(x)}{f(x)} x$

$$T(x \rightarrow 0) = C_1 + C_2 e^{u(x \rightarrow 0) * 0} \approx 600$$

$$C_1 + C_2 = 600$$

$$C_1 = 600 - \ C_2$$

The second boundary condition that we can use is that we can assume that the temperature at the interface between the tip and the cylinder at x = 50 cm is the same temperature of the surface of the cylinder T = 550 $°C$.

$$T(0.5) = 600 - \ C_2 + C_2 e^{u(0.5)} = 550$$

$$C_2 (e^{\frac{12.517}{2}} - 1) = -50$$

$$C_2 = -\frac{50}{e^{\frac{12.517}{2}} - 1} = -0.0959$$

The temperature profile at the tip can be expressed as:

$$T(x) = 599.9 - 0.0959e^{u(x)}$$

To model the second section, we can use the equation derived earlier:

$$T(x,r) = T_{min} + (T_{\max} - T_{min}) \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2) j_0 \left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right) j_0(j_{0,n} \frac{r}{r_{max}})$$

the slab can be modelled by the following equation:

$$\frac{\partial^2 T_x}{\partial x^2} + \frac{\partial^2 T_y}{\partial y^2} + \frac{\partial^2 T_z}{\partial z^2} = \nabla^2 T_{xyz} = 0$$

since the sides of the slab are insulated, we can assume that there is no temperature gradient in the y and z direction, therefore our governing equation becomes:

$$\frac{\partial^2 T_x}{\partial x^2} = 0$$

this PDE can be solved via direct integration:

$$\frac{d}{dx}\left(\frac{dT_x}{dx}\right) = 0$$

$$\int d\left(\frac{dT_x}{dx}\right) = \int 0 dx$$

$$\frac{dT_x}{dx} = C_1$$

$$\int dT_x = \int C_1 dx$$

$$T_x = C_1 x + C_2$$

applying the boundary conditions, we know that the temperature at the top of the slab is equal to the temperature at the bottom of the hot side and we assumed that the temperature of the surface of the slab is equal to 500 $°C$. As such we can form the following systems of equation

$$550 = C_1(250/100) + C_2$$

$$500 = C_1(350/100) + C_2$$

solving for the constants C_1 and C_2

$$C_1 = -50$$

$$C_2 = 675$$

$$T(x) = 675 - 50x$$

Since the solutions are somewhat discontinuous, we can try to model the temperature profile as follows:

$$T(x) = \begin{cases} x \in (0,0.5]; \quad 599.9 - 0.0959e^{u(x)} \\ x \in (0.5, 2.5):\ T_{min} + (T_{\max} - T_{min}) \sum_{n=0}^{\infty} \frac{1}{sin(j_{mn}^2) j_0 \left(\frac{j_{0,n}}{2}\right)} sin\left(-\frac{j_{mn}^2 x}{L}\right) j_0 (j_{0,n} \frac{r}{r_{max}}) \\ x \in (2.5, 3.5): 675 - 50x \end{cases}$$

for the complex problem we will use the same formula that we used to calculate the heat in however the thermal resistance can be calculated differently for each section.

for the cylindric rod we can use the following thermal resistance equation:

$$R_{cylinder} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k L}$$

and for the slab we can use the following:

$$R_{slab} = \sum \left(\frac{\Delta x}{kA}\right)_i$$

To calculate the heat supplied to the system we can use the general form of Fourier's law:

$$q = -k\nabla T_{x,r}$$

for this problem we will only consider the heat moving to the x direction this is because radial heating is only occurring in the second section so this can be calculated separately, the derivative is evaluated at r_max and x=0.5 since this is the hottest point of the cylinder:

$$q_x = -k\frac{dT}{dx}$$

$$T(x) = \begin{cases} 599.9 - 0.0959e^{u(x)} \\ T_{min} + (T_{\max} - T_{min}) \sum_{n=0}^{\infty} \frac{1}{sin(L^2) j_0 \left(\frac{j_{0,n}}{2}\right)} sin(-Lx) j_0 (j_{0,n} \frac{r}{r_{max}}) \\ 675 - 50x \end{cases} \rightarrow \frac{dT}{dx}|_{r=r_{max}}$$

$$= \begin{cases} -0.0959e^{u(0.5)} \\ L(T_{min} - T_{max}) \sum_{n=0}^{\infty} \frac{1}{sin(L^2) j_0 \left(\frac{j_{0,n}}{2}\right)} cos\left(-\frac{L}{2}\right) j_0 (j_{0,n}) \\ -50 \end{cases}$$

Since we are considering the heat that is being supplied from the hot end to the cold end we will not consider the contribution coming from the slab, to calculate therefore we can use the following:

$$q_x = -137\left(-0.0959e^{u(0.5)} - 0\right) = 6863.9 \approx 6.9\frac{kW}{m^2}$$

$$\dot{Q}_x = Aq_x = w_{slab}L_{slab}q_x = 0.5 * 1 * 6863.9 = 3431.95 \approx 3.4kW$$

To calculate the thermal resistance, since area is constant:

$$R_{slab} = \sum \left(\frac{\Delta x}{kA}\right)_i = \left(\frac{\Delta x}{kA}\right)_{macor} + \left(\frac{\Delta x}{kA}\right)_{aerogel} = \frac{1}{A}\left(\left(\frac{\Delta x}{k}\right)_{macor} + \left(\frac{\Delta x}{k}\right)_{aerogel}\right)$$

$$R_{slab} = 2 * \left(\frac{0.02}{1.45} + \frac{0.015}{0.023}\right) \approx 1.33\frac{K}{W}$$

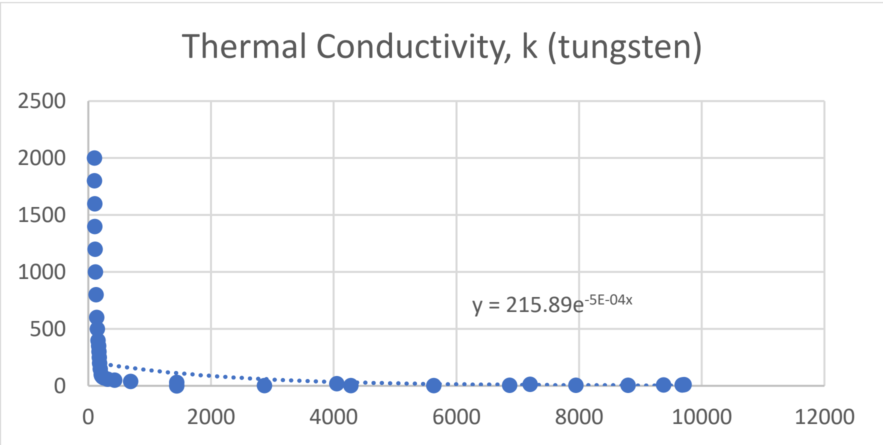
Calculating the cold temperature of the gas we get:

$$\dot{Q}_{in} = \frac{\mathrm{T_c} - T_h}{R_{thermal}} = 3431.95 * 1.33 = 600 - T_c$$

$$\%change = 100 * \left(\frac{4564.5}{500} - 1\right) = 812.9\%$$

Evaluation

The limitations to our comparison come from the various assumptions that we made throughout the portfolio as well as whether the geometry of the real engine was accurate enough to model a realistic behaviour. As such we can test some of our assumptions to evaluate whether they are adequate for our specific problem. The assumption that can have the largest impact on our results is that of constant thermal conductivity. The assumption that tungsten has constant thermal conductivity was tested by regressing data from [3].



The data was plotted over a large temperature range to observe global behaviour as well as how the thermal conductivity varies around our specific temperature range. Using an exponential regression, the data could be modelled as

$$k(T) = 215.89e^{-0.0005T}$$

This would make our integral to get the temperature profile from the heat equation vary. Therefore, this might not be an adequate assumption for large temperature changes below 1000 degrees as the exponential decays rapidly.

Another assumption made was that the tip of the cylinder would be placed in proximity to the focal point without considering the thermal resistance of the material. The resistance through the tip of the Stirling engine can be derived from Fourier's law:

$$\dot{Q}_{x,tip} = -kx^2\frac{dT}{dx}$$

we know that heat flow can be expressed as the ratio between the driving force (temperature difference) over the thermal resistance consequently, we can calculate the thermal resistance by integrating Fourier's Law to get the ΔT term:

$$\int_{x_{bottom}}^{x_{top}} \dot{Q}_{x,tip} x^{-2}\, dx = \int_{T_1}^{T_2} -kdT$$

assuming that thermal conductivity is constant over the temperature range integrated we get:

$$-\dot{Q}_{x,tip} \Delta x^{-1} = -k(T_2 - T_1)$$

$$\dot{Q}_{x,tip} = k(T_2 - T_1)\Delta x = \frac{\Delta T}{R_{tip}}$$

$$\therefore$$

$$R_{tip} = [k\Delta x]^{-1}$$

calculating this knowing that the tip is 50 cm long and is made of tungsten we get:

$$R_{tip} = \left(\frac{137}{2}\right)^{-1} \approx 0.0146$$

this is a small value as such the assumption can be deemed inadequate.

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Contents

Transport phenomena appendix

Kesler Isoko

November 2021

1 Nomeclature

1. $x_{contact}$ the lenght from the focal point to where the small and large rod touch
2. k_{hot} is the thermal conductivity of the hot material which in this case is tungsten
3. $(2\pi rL)_{cold}$ is the area of the cold side
4. $J(\lambda x)$ refers to the bessel function of x with eigenvalue λ
5. J_{mn} refers to the roots of the bessel function, see appendix section 3.3
6. r_o refers to the outer radius of the rod
7. r_i refers to the inner radius of the rod
8. T_c refers to the temperature of the cold side
9. T_h refers to the temperature of the hot side
10. r_{max} is the radius of the rod
11. L is the lenght of the rod
12. $\delta_{aerogel}$ refers to the thickness of the aerogel layer
13. R_{tot} refers to the total thermal resistance

14. R_{slab} refers to the thermal resistance of the slab

15. R_{tip} refers to the thermal resistance of the tip of the Stirling Engine

2 Introduction

2.1 Context

2.2 Configuration

2.3 The Model

3 The core problem

3.1 Simplifying the Geometry

3.2 Assumptions

3.3 Deriving the temperature profile

3.4 Evaluation

4 The complex problem

4.1 Modelling the geometry

4.2 Assumptions

4.3 Deriving temperature profile

4.4 Evaluation

5 Source Code

```

def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])
def approximate_bessel(x):
    return sp.j0(x)
def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x/L)*calculate_bessel(root(n)*r/r_max)
T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x/L)*approximate_bessel(root(n)*r/r_max)
x = np.linspace(0,L)
r = np.linspace(0,r_max)
x, r = np.meshgrid(x, r)
temperature = T(x,r,n=1)
df = pd.DataFrame(temperature)
for n in range(1, 301):
    temp = T(x,r,n)
    dataframe = pd.DataFrame(temp)
    for i in range(len(df)):
        df[i] += dataframe[i]
    if n == 1:
        ax = sb.heatmap(df, cmap=cm.hot)
        ax.set_xlabel('Distance From Focal Point, x (cm)')
        ax.set_ylabel('Radius, r (cm)')
        ax.set_title = '{n} terms'
        ax.invert_yaxis()
        ax.plot()
        plt.show()
    elif n == 100:
        ax = sb.heatmap(df, cmap=cm.hot)
        ax.set_xlabel('Distance From Focal Point, x (cm)')
        ax.set_ylabel('Radius, r (cm)')
        ax.set_title = '{n} terms'
        ax.invert_yaxis()
        ax.plot()
        plt.show()
    elif n == 200:
        ax = sb.heatmap(df, cmap=cm.hot)
        ax.set_xlabel('Distance From Focal Point, x (cm)')
        ax.set_ylabel('Radius, r (cm)')
        ax.invert_yaxis()
        ax.set_title = '{n} terms'
        ax.plot()
        plt.show()
    elif n == 300:
        ax = sb.heatmap(df, cmap=cm.hot)
        ax.set_xlabel('Distance From Focal Point, x (cm)')
        ax.set_ylabel('Radius, r (cm)')
        ax.invert_yaxis()
```

```python
        ax.set_title = '{n} terms'
        ax.plot()
        plt.show()
    else:
        pass

import numpy as np
from matplotlib import pyplot as plt
import mpmath
import pandas as pd

t_max = 600
t_min = 100
r_max = 35
L = 350

def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])

def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x/L)*calculate_bessel(root(n)*r/r_max)
x = np.linspace(0,L)
r = np.linspace(0,r_max)
temperature = T(x,r,n=1)
df = pd.DataFrame(temperature)
result = pd.DataFrame(temperature)
for n in range(1,300):
    temperature = T(x,r,n)
    dataframe = pd.DataFrame(temperature)
    df[0] += dataframe[0]
    if n == 5:
        result[f'{n} terms'] = df[0]
    elif n == 25:
         result[f'{n} terms'] = df[0]
    elif n == 65:
         result[f'{n} terms'] = df[0]
    elif n == 95:
         result[f'{n} terms'] = df[0]
    elif n == 125:
         result[f'{n} terms'] = df[0]
    elif n == 165:
         result[f'{n} terms'] = df[0]
    elif n == 195:
         result[f'{n} terms'] = df[0]
    elif n == 225:
        result[f'{n} terms'] = df[0]
    elif n == 265:
         result[f'{n} terms'] = df[0]
    elif n == 295:
         result[f'{n} terms'] = df[0]

```

```
    else:
        pass

result.plot()
result.to_csv('temp profile core.csv')
plt.ylabel('Temperature T ( C )')
plt.xlabel('Distance From Focal Point , x (cm)')
plt.show()

import numpy as np
from matplotlib import pyplot as plt
import mpmath
import pandas as pd


df = pd.DataFrame([float(str(mpmath.besselj(0,x))) for x in range(50)])
df['J0(x)'] = pd.DataFrame([float(str(mpmath.besselj(0,x))) for x in range(50)])
df.drop([0],axis=1,inplace=True)
for i in range(1,3):
    df[f'J{i}(x)'] = pd.DataFrame([float(str(mpmath.besselj(i,x))) for x in range(50)])
print(mpmath.besseljzero(0,1))
print(df)
df.plot()
plt.ylabel('Bessel Function')
plt.xlabel('x')
plt.axhline(0.0,color='black')
plt.show()

from matplotlib import cm
from matplotlib import pyplot as plt
import mpmath
import pandas as pd
import seaborn as sb
from scipy import special as sp
import numpy as np
t_max = 600
t_min = 100
r_max = 35
L = 350

def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])
def approximate_bessel(x):
    return sp.j0(x)
def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x/L)*approximate_bessel(root(n)*r/r_max)
x = np.linspace(0,L)
r = np.linspace(0,r_max)
```

```
x, r = np.meshgrid(x, r)
temperature = T(x,r,n=1)
dataframe = pd.DataFrame(temperature)
for i in range(len(dataframe)):
    dataframe[f'x = {i} (cm)'] = dataframe[i]
dataframe.drop([i for i in range(len(dataframe))],axis=1,inplace=True)
dataframe[[f'x = {i} (cm)' for i in range(14)]].plot()
plt.xlabel('Distance From Focal Point, x (cm)')
plt.ylabel('Temperature, T ( C )')
plt.show()

import numpy as np
from matplotlib import pyplot as plt
from matplotlib import cm
import mpmath
import pandas as pd
import seaborn as sb
from scipy import special as sp
t_max = 600
t_min = 100
r_max = 35
L = 350

def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])
def approximate_bessel(x):
    return sp.j0(x)
def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x/L)*approximate_bessel(root(n)*r/r_max)
x = np.linspace(0,L)
r = np.linspace(0,r_max)
x, r = np.meshgrid(x, r)
temperature = T(x,r,n=1)
fig = plt.figure(figsize = [12,8])
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(x, r, temperature, cmap=cm.coolwarm)
ax.set_xlabel('Distance From Focal Point, x (cm)')
ax.set_ylabel('Radius, r (cm)')
ax.set_zlabel('Temperature T ( C )')
plt.show()

import numpy as np
from matplotlib import pyplot as plt
from matplotlib import cm
import mpmath
import pandas as pd
import seaborn as sb
from scipy import special as sp
t_max = 600
t_min = 100
```

```
r_max = 35
L = 350

def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])
def approximate_bessel(x):
    return sp.j0(x)
def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

def F(x):
    try:
        return ((4/9)*x**11-(8/7)*x**9+(2/5)*x*7+(1/3)*x**5)/((44/9)*x**10-(72/7)*x**8+(14/5)*x**6+(5/3)*x**4)
    except ZeroDivisionError:
        x = 0.00000000001
        return ((4/9)*x**11-(8/7)*x**9+(2/5)*x*7+(1/3)*x**5)/((44/9)*x**10-(72/7)*x**8+(14/5)*x**6+(5/3)*x**4)

def G(x):
    return 675 -50*x

x = [i for i in range(L)]
r = r_max
n = 1
T_x = []
for x_i in x:
    if x_i < 50:
        T_x.append(599.9 + 0.0959*np.e**(-F(x_i)*x_i))
    elif x_i < 100 and x_i > 50:
        T_x.append(t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)**2)*x_i/L)*approximate_bessel(root(n)*r/r_max))
    else:
        T_x.append(G(x_i))

#df = pd.read_csv('temp profile core.csv')
plt.style.use('fivethirtyeight')
plt.legend('@ Radius = 35 (cm)')
plt.xlabel('Distance From Focal Point, x (cm)')
plt.ylabel('Temperature, T ( C )')
plt.title('Temperature Profile (1st Rod)')
plt.plot(x, T_x)
plt.show()

import numpy as np
from matplotlib import pyplot as plt
import mpmath
import pandas as pd

t_max = 600
t_min = 100
r_max = 35
L = 350

```

```python
def calculate_bessel(x):
    return np.array([float(str(mpmath.besselj(0,x))) for x in x])

def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def B(n):
    return 1/np.sin(root(n)**2)

T = lambda x,r, n: t_min + (t_max - t_min)*B(n)*np.sin(-(root(n)
    **2)*x/L)*calculate_bessel(root(n)*r/r_max)
x = np.linspace(0,L)
r = np.linspace(0,r_max)
temperature = T(x,r,n=1)
plt.style.use('fivethirtyeight')
plt.legend('@ Radius = 35 (cm)')
plt.xlabel('Distance From Focal Point, x (cm)')
plt.ylabel('Temperature, T ( C )')
plt.title('Temperature Profile (Core)')
plt.plot(x,temperature)
plt.show()
```

Source Code 1: The code used to plot the different graphs

```python
n = 1
x = 2.5
r = 0.35
k_tungsten = 137
L_smallrod = L - 1

def calculate_bessel(x):
    return float(str(mpmath.besselj(0,x)))

def root(n):
    return float(str(mpmath.besseljzero(0,n)))

def factorial(n):
    x = 1
    for i in range(n):
        x *= (n - i)
    return x

beta = (root(n)**2)/L

def B(n):
    return 1/np.sin(root(n)**2)

dQ_r = (t_max - t_min)*(B(n)*np.sin(-beta*x)*(root(n)/2)*(n/r_max
    )*(((-1)**n)/(factorial(n)*math.gamma(n+1)))*(((root(n)/2)*(r
    /r_max))**(2*n-1)))
print(dQ_r)

dQ_x = ((root(n)**2)/L)*(t_min - t_max)*(B(n)*np.cos(-beta)*x)*
    calculate_bessel(root(n)*(r/r_max))
print(dQ_x)

Q1 = -2*np.pi*r*L_smallrod*k_tungsten*dQ_r
print(Q1)
```

```
Q2 = -2*np.pi*r*L_smallrod*k_tungsten*dQ_x
print(Q2)

change = Q2*0.215

percentage_change = (change/500 - 1)*100
print(percentage_change)
```

Source Code 2: The code used to calculate the percentage change in the core problemlabel

Source Code